Witten Nucl Phys B160 (1979): QCD large-N expansion vs. QED

Just for the sake of comparison, let us ask why perturbation theory is successful in QED. It is not enough to say that "the electric charge is small." In fact, normalized in the usual way so that the interaction vertex is just $e\gamma^\mu$, the electric charge is approximately $e \doteq 0.302$. Perturbation theory is a good approximation in QED because when one carries out perturbative expansions, one finds that the typi-

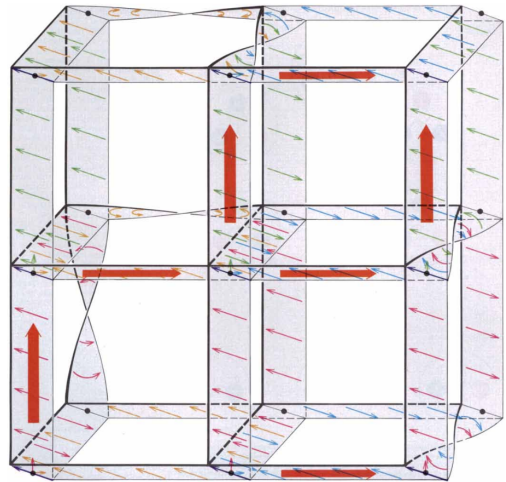
82 *E. Witten / Baryons in the 1/N expansion*

cal expansion parameter is really $e^2/4\pi$. If the typical parameter had turned out to be $4\pi e^2$, perturbation theory would not have been very successful for e as large as 0.302. And if we had not yet learned how to do perturbative calculations, we would have been unable to judge, just from the fact $e = 0.302$, whether an expansion in e^2 would be a good approximation.

If, for instance, as is perfectly possible, the characteristic parameter in the non-planar diagrams is really not $1/N^2$ but $1/4\pi N^2 = 1/113$, then non-planar diagrams in QCD are almost as tiny as electromagnetic corrections. While this is only an extreme possibility, there is no reason to be surprised that phenomenology seems to show that the $1/N$ expansion is a good approximation.

Lattice Gauge Theory of QCD cartoon

- consider Feynman path integrals, sum over all possible paths between two times
- Spacetime discretized:
  Quarks on vertices.
  Gluons travel along lines between vertices
  m_q taken nonphysically large (expressed as large m_π)
  Want lattice spacing $\rightarrow 0$, times, volume $\rightarrow \infty$, $m \rightarrow$ physical
- Numerical results include
  quark confinement,
  m_{hadrons},
  axial vector coupling g_A of n,
  π decay constant f_π



Rebbi Sci Am 248 54 (1983)

Isospin in Nuclei: examples

- **Isospin and two-nucleon states (Deuteron and np, pp, nn resonances)**
- **Some implications of isospin, spin, permutation symmetry (Wong; EGA Phys 562 UW):**
  - **μ contribution from deuteron's D-state admixture: calculation**
  - **No π exchange currents in μ_{deuteron}**
    - **(and isospin of γ)**
- **Isobaric analog states: Wong's example of relative energies of A=16**
- **Isospin Multiplet Mass Equation for hadrons**
  IMME for nuclei
- **Isospin breaking example**

Reminder: Properties of the deuteron and np system

n–p bound system: the 'hydrogen atom' of nuclear physics

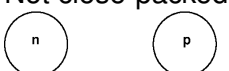
there is only 1 bound state !

$\sqrt{< r_d^2 >} \approx 2.1$ fm (electron scattering)

$\sqrt{< r_p^2 >} = 0.890 \pm 0.014$ fm

(Garching hydrogen lamb shift 1997; CJP 77 241 (1999)) (4% smaller now ☺)

Not close-packed like A>2, cartoon looks like:



Force between nucleons is in some sense not that strong

$J^\pi = 1^+$

μ=0.8574376$\pm$0.0000004 μ_N

Q=2.88 $\pm$ 0.002 barn

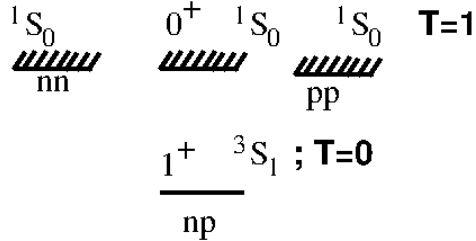
B.E. = 2.22463 $\pm$ 0.00004 MeV

2-Nucleon states and isospin: n,p 'identical'

- np has a scattering resonance 70 keV above zero energy,
- pp has scattering enhancement at low E (if subtract Coulomb, a_{pp} close to a_{np})
- nn and pp have T_3 = +1 and -1, so they have total isospin $T = 1$, symmetric
- Best guess: low L (not J!) has lower energy,
- wf must satisfy antisymmetry under fermion exchange: $\psi(-) = \psi_T(+)\psi_{\rm space}(+)\psi_{\rm spin}(-)$ so L=0 (parity even $\Rightarrow \psi_{\rm space}(+)$

Leaving possibilities:

- Isospin triplet of unbound resonances: $\psi_{T=1}(+)\psi_{S=0}(-)$
- ground state bound by 2.25 MeV $\psi_{T=0}(-)\psi_{S=1}(+)$



- Isobaric analog states have almost same nuclear wavefunctions, to $\sim$ 1% accuracy
- Useful in nuclear β decay, because the nuclear matrix elements needed are very simple to calculate
- Odd-Odd N=Z have these deuteron-like states

deuteron magnetic moment with D-state Wong ¶3-2

let the deuteron wavefunction have an I=2 component.
We still need the intrinsic spin S=1 so that it is symmetric (because L is still even and therefore space is symmetric, and T=0 is antisymmetric). So:

$$|\psi_d\rangle = \sqrt{1-\epsilon^2}|^3S_1\rangle + \epsilon|^3D_1\rangle$$

μ only has effect on S and L, so there at no S$\times$D cross-terms involving S times D, because the spatial wavefunctions are Y_{LM}'s and they are orthogonal.
So we just need to evaluate μ separately in the S and D pieces.
For L, for a charged particle, orbital angular momentum creates a current loop,

$$\vec{\mu}_{\rm orbital} = \frac{e\hbar c}{2M_p c}\vec{l}$$

or in terms of gyromagnetic ratio

$$\vec{\mu}_{\rm orbital} = g_l\vec{l}$$

g_l=μ_N for proton, "1" in these units; $g_l = 0$ for neutron because it has no charge

then $\vec{\mu}_d = g_p\vec{s}_p + g_n\vec{s}_n + \vec{l}_p$

Since masses of p, n are $\approx$ equal, $\vec{l}_p$ = 1/2 $\vec{L}$ (total deuteron angular momentum)

Rewrite

$$g_p\vec{s}_p + g_n\vec{s}_n = \frac{1}{2}\left((g_p + g_n)(\vec{s}_p + \vec{s}_n) + (g_p - g_n)(\vec{s}_p - \vec{s}_n)\right)$$

$\vec{s_p} - \vec{s_n}$ acts on proton and neutron spins with opposite sign, so will vanish for states with same S=0 or S=1 (and ours is S=1).

So finally can write the operator in terms of total angular momenta:

$\mu_d = \frac{1}{2}\left((g_p + g_n)\vec{S} + \vec{L}\right)$;

If we use some advanced vector operator concepts, we avoid writing down explicit wavefunctions and angular integrals. (Consult Wong on Wigner-Eckhart theorem)

The magnetic moment is given by the z-projection of μ, which in spherical tensor notation means

$\mu = \langle J, M = J|\mu_0|J, M = J\rangle$

μ_z and J_z are both z-projections of vector operators, so they are proportional to each other. The proportion is given by Landé formula, which we now take from Wong's appendix A:

A-6 Landé Formula

Consider a vector operator $\boldsymbol{V}$. Since it is an operator with spherical tensor rank unity, its matrix element behaves, under a rotation of the coordinate system, in the same way as any other spherical tensor of the same rank, including the angular momentum operator $\boldsymbol{J}$. Using the Wigner-Eckart theorem, the matrix element of component q of $\boldsymbol{V}$ may be expressed in terms of its reduced matrix element as

$$\langle JM|V_{1q}|JM'\rangle = (-1)^{J-M}\begin{pmatrix} J & 1 & J \\ -M & q & M' \end{pmatrix}\langle J\|\boldsymbol{V}\|J\rangle \tag{A-16}$$

where q has possible values ± 1 and 0. Similarly, the matrix element of $\boldsymbol{J}$ has the form

$$\langle JM|J_{1q}|JM'\rangle = (-1)^{J-M}\begin{pmatrix} J & 1 & J \\ -M & q & M' \end{pmatrix}\langle J\|\boldsymbol{J}\|J\rangle \tag{A-17}$$

Since both reduced matrix elements $\langle J\|\boldsymbol{V}\|J\rangle$ and $\langle J\|\boldsymbol{J}\|J\rangle$ are quantities independent of the coordinate system, they must be multiples of each other, with the ratio

$$\mathcal{R} = \frac{\langle J\|\boldsymbol{V}\|J\rangle}{\langle J\|\boldsymbol{J}\|J\rangle} \tag{A-18}$$

independent of M.

We can check that this is the same as scalar products defined in terms of Cartesian components of the vectors by noting that

$$\begin{aligned} \boldsymbol{J}_{\pm 1} &= \mp\frac{1}{\sqrt{2}}(\boldsymbol{J}_x \pm i\boldsymbol{J}_y) & \boldsymbol{J}_0 &= \boldsymbol{J}_z \\ \boldsymbol{V}_{\pm 1} &= \mp\frac{1}{\sqrt{2}}(\boldsymbol{V}_x \pm i\boldsymbol{V}_y) & \boldsymbol{V}_0 &= \boldsymbol{V}_z \end{aligned}$$

This is slightly different from the definition of angular momentum raising and lowering operators $\boldsymbol{L}_\pm = \boldsymbol{L}_x \pm i\boldsymbol{L}_y$ as the usual convention does not attempt to make them spherical tensor operators.

We can now make an intermediate state expansion of the matrix element of $\boldsymbol{J}\cdot\boldsymbol{V}$,

$$\langle JM|(\boldsymbol{J}\cdot\boldsymbol{V})|JM\rangle = \sum_{M'}\sum_{q}(-1)^q\langle JM|\boldsymbol{J}_q|JM'\rangle\langle JM'|\boldsymbol{V}_{-q}|JM\rangle$$

Since the operator $\boldsymbol{J}$ can change at most the M-value, but not the J-value, of a function on which it acts, a sum over intermediate states of different J-values is not needed. Using the ratio $\mathcal{R}$ defined in Eq. (A-18) and the relations given by Eq. (A-16) and Eq. (A-17), we have the relation

$$\begin{aligned} \langle JM'|\boldsymbol{V}_{-q}|JM\rangle &= (-1)^{J-M'}\begin{pmatrix} J & 1 & J \\ -M' & -q & M \end{pmatrix}\langle J\|\boldsymbol{V}\|J\rangle \\ &= (-1)^{J-M'}\begin{pmatrix} J & 1 & J \\ -M' & -q & M \end{pmatrix}\mathcal{R}\langle J\|\boldsymbol{J}\|J\rangle \\ &= \mathcal{R}\langle JM'|\boldsymbol{J}_{-q}|JM\rangle \end{aligned}$$

With this, we obtain the result

$$\begin{aligned} \langle JM|(\boldsymbol{J}\cdot\boldsymbol{V})|JM\rangle &= \mathcal{R}\sum_{M'}\sum_{q}(-1)^q\langle JM|\boldsymbol{J}_q|JM'\rangle\langle JM'|\boldsymbol{J}_{-q}|JM\rangle \\ &= \mathcal{R}\langle JM|\boldsymbol{J}^2|JM\rangle \\ &= \mathcal{R}J(J+1) \end{aligned}$$

In other words, $\mathcal{R} = \langle JM|(\boldsymbol{J}\cdot\boldsymbol{V})|JM\rangle/J(J+1)$ and

$$\langle JM|V_q|JM'\rangle = \frac{1}{J(J+1)}\langle JM|(\boldsymbol{J}\cdot\boldsymbol{V})|JM\rangle\langle JM|\boldsymbol{J}_q|JM'\rangle$$

generally known as the Landé formula.

$$\langle J, M|\mu_0|J, M\rangle = \frac{1}{J(J+1)}\langle J, M|(\vec{\mu}\cdot\vec{J})J_0|J, M\rangle = \frac{M}{J(J+1)}\langle J, M|(\vec{\mu}\cdot\vec{J})|J, M\rangle$$

which then gives

$$\langle J, M|\mu_0|J, M\rangle = \frac{M}{J(J+1)}\langle J, M|\frac{1}{2}\left((g_p+g_n)\vec{S}\cdot\vec{J}+\vec{L}\cdot\vec{J}\right)|J, M\rangle$$

Then using the standard trick to get the dot products in terms of numbers that we know for each state:

$$\vec{S} \cdot \vec{J} = \vec{S} \cdot (\vec{L} + \vec{S}) = \vec{S}^2 + \frac{1}{2}(\vec{J}^2 - \vec{L}^2 - \vec{S}^2)$$

and similarly

$$\vec{L} \cdot \vec{J} = \frac{1}{2}(\vec{J}^2 + \vec{L}^2 - \vec{S}^2)$$

we get

$$\mu_d = \frac{1}{4(J+1)}((g_p + g_n)(J(J+1) - L(L+1) + S(S+1)) + (J(J+1) + L(L+1) - S(S+1)))$$

So for $^3S_1$ pure s-state, i.e. L=0, S=1, J=1, we get recover the simple sum $\mu_d = \mu_p + \mu_n$ as before, i.e. the nucleon spins are just fully aligned to total spin 1.

For $^3D_1$, i.e. L=2, we get

$$\mu_d(^3D_1) = \frac{1}{8}((g_p + g_n)(-2) + 6) = 0.310\mu_N$$

and

$$\mu_d = \sqrt{1-\epsilon^2}\mu_d(^3S_1) + \epsilon^2\mu_d(^3D_1) = 0.857\mu_N$$

which gives $\epsilon^2 \approx 0.04$, the number mentioned last Wednesday for the D-state admixture.
Or one could also say ϵ=0.16, which is not so small, for operators/observables that can mix the terms:

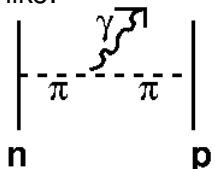
$$|\psi_d\rangle = \sqrt{1-\epsilon^2}|^3S_1\rangle + \epsilon|^3D_1\rangle$$

The atomic 'hyperfine anomaly,' which involves the next-order nuclear magnetic moment beyond dipole, is actually one such observable... the relative hyperfine splittings of hydrogen, deuterium, and tritium depend on the nuclear structure, and is consistently understood to 3% (Friar and Payne, Phys Lett B 618 (2005) 68).

π exchange currents do not contribute to μ_d (EGA UW PH562)

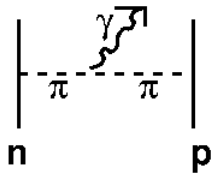
"meson exchange currents": Although the π's are virtual, the virtual presence of that charge can still in principle contribute to observables.
(One main operator in β decay ends up proportional to only electric charge, the 'conserved vector current' hypothesis, but a second operator does not...)

A diagram looks like:



1st we show the photon γ has isovector and isoscalar parts:
Photon couples to electromagnetic charge $\boldsymbol{q}$
For anything in first generation of particles (anything made of u, d quarks):
$\boldsymbol{q} = \boldsymbol{T_3} + \boldsymbol{B}/\mathbf{2}$ where B is baryon number.
(for $\boldsymbol{u}$, $\boldsymbol{T_3}$=+1/2, $\boldsymbol{B}$=1/3, q=2/3 Q.E.D.
for $\boldsymbol{d}$, $\boldsymbol{T_3}$ =-1/2, $\boldsymbol{B}$=1/3, q=-1/3)
$\boldsymbol{T_3}$ is an isospin vector.
$\boldsymbol{B}$ doesn't care about isospin, so it's an isoscalar

Since the deuteron g.s. has T=0, $\langle \boldsymbol{d}|\gamma|\boldsymbol{d}\rangle = 0$ for the $\boldsymbol{\Delta}$T=1 isovector piece, so only $\boldsymbol{\Delta}$T=0 piece of photon contributes $\rightarrow$



J^π of photon is 1^- (ignoring orbital angular momentum)
So the J^π of the two pions at the vertex must also together be 1^-
Because π is a boson,
the two-π state must have symmetric wf under exchange

π g.s. has S=0, so the two π's need to together have $\vec{J}$=1.
That must be orbital $\vec{L}$=1, which is always antisymmetric.
So to get a symmetric total wf, the isospin wf is also antisymmetric.

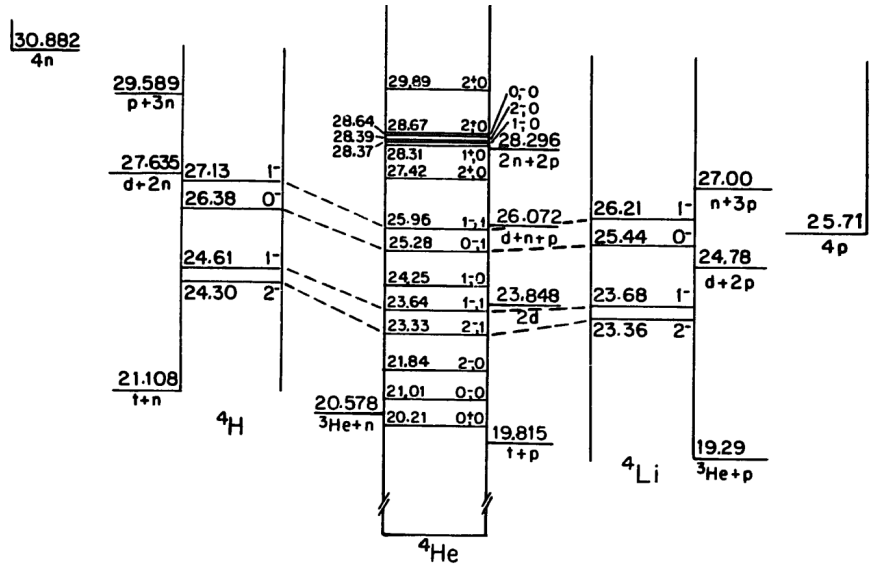
T_π=1, so two pions can have T=2, 1, or 0; of these, only T=1 is antisymmetric
Therefore the γ for this case must also have isospin =1
But we saw above that piece does not contribute
So π exchange currents don't contribute to μ_d
The D-state probability of 0.04 seems accepted as the reason for $\mu_d \neq \mu_n+\mu_p$

A=4 isobaric analog states

Note: the only bound states for A=3 are g.s. of ^{3}H and ^{3}He; all excited states unbound to particle emission as well

Is the 'tetraneutron' bound?

All energies w.r.t. ^{4}He g.s.



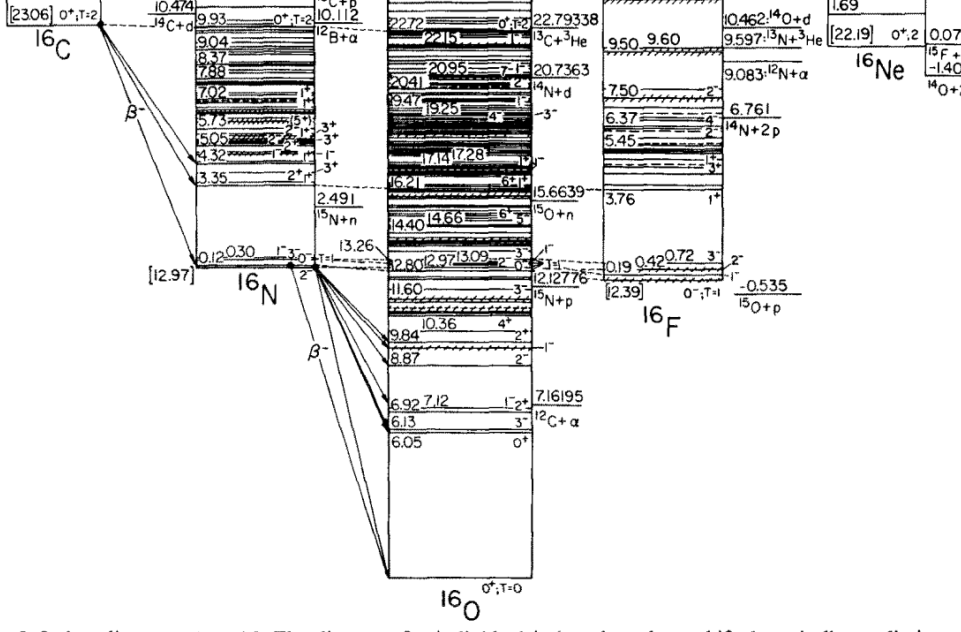


Fig. 5. Isobar diagram, $A = 16$. The diagrams for individual isobars have been shifted vertically to eliminate the neutron–proton mass difference and the Coulomb energy, taken as $E_C = 0.60Z(Z-1)/A^{1/3}$ MeV. Energies in

Energy of isobaric analog of ^{16}N in ^{16}O (Wong p.138)

If all forces were charge independent, the state in ^{16}O would have same mass, with excitation energy $E_B(^{16}\text{O})$-$E_B(^{16}\text{N})$=127.62-117.98=9.64 MeV

Two corrections:

- **Coulomb energy E_C for a uniform charged sphere: 3/5 $\alpha\ \hbar$c Z(Z-1)/R so difference is 6/5(Z-1)$\alpha\hbar$c/R**
  R$\approx$1.2 A$^{1/3}$ fm = 3.02 fm, $\alpha\hbar$c= 1.44 MeV-fm
  ΔE_C=4.00 MeV (This is pretty big)
- **p-n mass difference=.78 MeV**
- **Predicted Result: 12.86 MeV**
  There is a 2$-$, T=1 state at 12.97 MeV in ^{16}O.
  Off by 0.11 MeV. Not bad, 2% off in the shift in some sense.
  Why the difference? $\rightarrow$

Note: the four states near the g.s. of ^{16}N all have analogs in ^{16}O, and they are not in the same energy order. Our Coulomb energy estimate is too simple (should use real wavefunctions) and the wavefunctions in ^{16}N and ^{16}O are not exactly the same. E.g., the proton and neutron wavefunctions are not the same, because the proton wavefunction sticks out: this is the 'Thomas-Ehrman shift'.
Even so, the energy differences of isobaric analogs are not fully understood. This is known as the 'Nolen-Schiffer anomaly'. ΔE_C after real corrections is found to be 7% smaller than experimental values. After several small traditional corrections, modern approaches (chiral symmetry restoration in finite nuclei Henley 62 2586 (1989); QCD sum rules T. Hatsuda PRL 66, 2851 (1991); density-dependent charge symmetry Horowitz PRC 63, 011303 (2001); N. Kaiser PRCC 69, 034337 (2004) relativistic mean-field theory Agrawal PRC 64 024306 (2001); neutron skin (J. Duflo PRC 66 051304 (2002)) could contribute. Enough free parameters to fit, but explanations redundant and so not differentiated. We understand the Coulomb interaction, so we can constrain models.

Detailed density functionals tuned to fix the N-S anomaly need more isospin breaking in the strong interaction than expected. This approach leads to different isospin mixing in cases important to standard model tests in nuclear beta decay (Koniezcka Baczyk Satula 1909.09350).

Isobaric multiplet mass equation: particles

Again, from electric charge $q = T_3 + B/2$, can see isovector and isoscalar contributions to Coulomb interaction

A wf's perturbation in 1st-order perturbation theory is linear with the interaction $\frac{|i\rangle\langle i|H_{\rm Coul}|j\rangle}{\Delta E}$, while the energy shift is quadratic $\sim \frac{|\langle i|H_{\rm Coul}|j\rangle|^2}{\Delta E}$

$\Rightarrow$ from q^2, there's an isotensor added.

So for isobaric analog states, treating Coulomb interaction in 1st-order perturbation theory gives their relative masses (binding energies):

$M(T_3) = a + bT_3 + cT_3^2$

Idea: test such relations, independent of tough calculations for a, b, c

Weinberg and Treiman Phys Rev 116 465 (1959)

for π, b =0, c = 4.6 MeV = $m_{\pi\pm} - m_{\pi 0}$

They also applied to nucleon, K, Σ, but complained were limited to $T_3 \leq 3/2$ so could not test the relation

They also suggested isobaric analogs in nuclei $\rightarrow$

Isobaric multiplet mass equation: nuclei

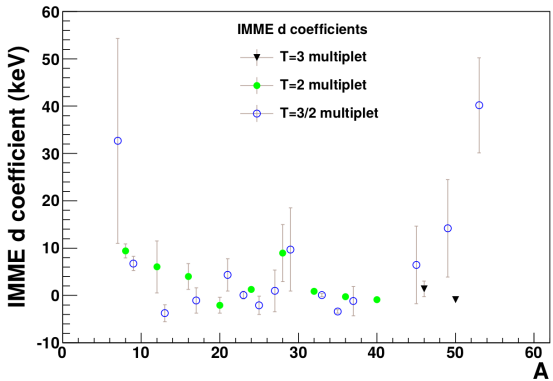
Jänecke Nucl Phys A128 632 (1969) taking again Coulomb only, but to 2nd order in perturbation theory,

$$M(T_3) = a + bT_3 + cT_3^2 + dT_3^3 + eT_3^4$$

where the coefficients are functions of A (i.e. Z) and, generally, T.

Thus the Coulomb interaction can produce a nonzero *d*, *e*. Nonzero *d*, *e* do not necessarily indicate more isospin breaking in the strong interaction– one needs to calculate a Coulomb correction. The Coulomb interaction is exactly known; wf's less so ☹

MacCormick+Audi, EPJWed 66 02065 (2014)

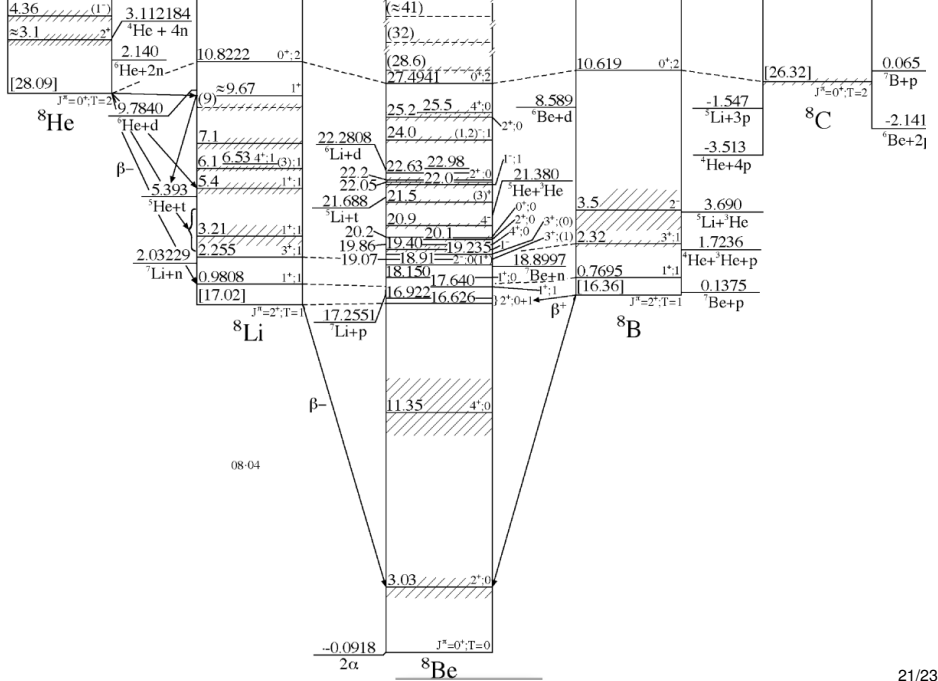


Precise A=9 example Brodeur et al PRL 108 212501 (2012)

d **=6.3± 1.7 might be from mixing with a nearby state of unknown J^π, i.e. likely no non-Coulomb isospin breaking needed**

A=8: How to measure the isospin of a state?

- T=1 states can't α-decay
- ^{10}B(d,α)^{8}Be only populates T=0 levels
- The 16.922,16.6262 pair are highly mixed and do not have good isospin



Simple model for isospin mixing in ^{8}Be

$|16.6\rangle = \cos\theta|T=0\rangle + \sin\theta|T=1\rangle$
$|16.9\rangle = -\sin\theta|T=0\rangle + \cos\theta|T=1\rangle$
Using first-order perturbation theory,
$\sin\theta \approx \frac{\langle T=1|H_{\rm Coul}|T=0\rangle}{(16.626-16.922)\ \rm MeV}$
$\frac{\Gamma_\alpha(16.9)}{\Gamma_\alpha(16.6)} = 0.69 \Rightarrow \theta \sim 40^o$
isospin means little

It turns out the wavefunctions need to be done well... because $H_{\rm Coul} \sim r^2$, in the end the isospin mixing physics weights the high-r tails of the wavefunctions.
But one can schematically write the main effect (McDonald and Adelberger PRL 40 1692 (1978)):

Consider the analog states ('π' means proton, 'ν' means neutron):
$|^8{\rm Li}\ T=1\rangle = |\pi(1p_{1/2})\nu(1p_{3/2})\rangle$
$|^8{\rm B}\ T=1\rangle = |\pi(1p_{3/2})\nu(1p_{1/2})\rangle$
The T=1 configuration has to be symmetric in isospin
$|T=1\rangle = |^8{\rm Li}\ T=1\rangle + |^8{\rm B}\ T=1\rangle$
The T=0 configuration is called the 'anti-analog' and is antisymmetric in isospin
$|T=0\rangle = |^8{\rm Li}\ T=1\rangle - |^8{\rm B}\ T=1\rangle$
The minus sign leads to cancellation: the matrix element is an order of magnitude smaller than the Coulomb energy of a charged sphere.

The isospin mixing of such analog/'antianalog' states was carefully studied experimentally in ^{8}Be, ^{12}C, ^{16}O, ^{24}Mg, etc.

At one time it was thought experimental matrix elements were more than a factor of two larger than could be accounted for theoretically from just the Coulomb interaction (Adelberger PhysRevC 15 484 (1977)),

so maybe were evidence for extra isospin violation in strong interaction.

More careful wavefunctions could account for it (Shlomo ZPhysA 285 283 (1978)),
though since then more accurate experiments have been done to test these (von Neumann-Cosel Nucl Phys A 669 3 (2000)).

Important to understand these effects, because they make small corrections to precision β decay tests of the standard model.