Nucleons and their Structure; Hadrons and Isospin

- **quark content of baryons and mesons**
- **Isospin symmetry and some consequences**
- **Antisymmetry of wavefunctions under exchange of fermions and examples**

μ of *p*, *n* in constituent quark model
$\Rightarrow$ "Color" (QCD 'charge') is a needed degree of freedom

- References:

Wong ch. 2
Halzen and Martin, Quarks and Leptons, Ch. 2: "HM"
EGA's lectures

quark content of n, p; isospin

ignoring antisymmeterization for now

$|p\rangle = |uud\rangle$ $|n\rangle = |udd\rangle$

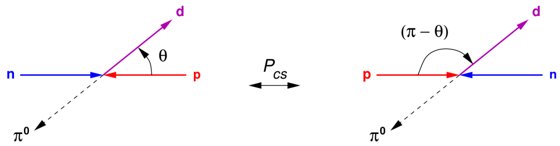
u **has electric charge q = 2/3,** ***d*** **has -1/3**

Consider the nucleon, a spin-1/2 fermion with isospin t**=1/2,**
and isospin projection along the quantization axis
$t_0 = t_3 = +1/2$ **for proton and -1/2 for the neutron**
Then for nuclei, isospin projection just counts protons and neutrons
$T_3 = (Z-N)/2$
But there is a lot more physics in the total isospin T
(We may switch to the typical nuclear sign convention later, but we often study Z$>$N ☺)

Heisenberg invented isospin (Zeit. für Physik 77 1 (1932)) more as a label than anything else, but it turns out to be a powerful symmetry in nuclear physics:

good Isospin symmetry: qualitative informal comments

This field was in progress for Wong 1999:
The strong interaction respects isospin symmetry well.



See e.g. Opper... Hutcheon... Yen et al. PRL 91 212302 (2003) charge symmetry breaking in front-back asymmetry of $n\, p \to d\, \pi^0$
$A_{fb} = 1.7 \pm 0.8 \pm 0.6 \times 10^{-3}$

Such effects all traceable to $m_u \neq m_d$ so far:
review Miller et al. Ann Rev Nucl Part Sci 56 253 (2006).

Coulomb and other E&M can usually be treated perturbatively.
Many interesting phenomena in nuclear structure and reactions from isospin symmetry: an active field testing the strong interaction into the 1980's

Isospin breaking: N-N scattering lengths, binding energies

- **n p mass difference is about 1 MeV out of 1000 MeV.**
- **p. 93 and 94 of Wong mention reasons for the higher value of a_{np} compared to a_{nn} and a_{pp} (e.g. in terms of mesons of different mass and $\therefore$ range). This is thought to be understood, and not breaking isospin symmetry in the strong interaction.**

Indirectly Chen et al PRC 77 054002 (2008) measured in $d(\pi^-, n\gamma)n$
a_{nn} = -18.9$\pm$0.4 fm
compared to a_{pp} = -17.3 $\pm$ 0.4 fm (raw value is 7.8063 fm, corrected for E&M).
The difference is 1.6$\pm$0.5 fm different from a_{pp}.

This effect is enhanced by 10 by the physics compared to A_{fb} (Gardestig JPG 36 (2009) 053001; Machleidt and Entem Phys Rep 503 (2011) 1), and it's all consistent.

(• The 'Nolen-Schiffer anomaly' in binding energies of T=1/2 nuclei is not necessarily understood yet ☺ ?)

Isospin symmetry: treat, as spin, like SU(2)

We can write the nucleon as a 2-component column matrix:

$$|p\rangle = |t{=}\tfrac{1}{2}, t_0{=}+\tfrac{1}{2}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_t$$

$$|n\rangle = |t{=}\tfrac{1}{2}, t_0{=}-\tfrac{1}{2}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_t$$

with formal operators like Pauli spin matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and commutation relation

$$\tau_i \tau_j = \delta_{ij} I + i\epsilon_{ijk}\tau_k$$

Construct raising and lowering operators:

$$\tau_+ = \tfrac{1}{2}(\tau_1 + i\tau_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \tau_- = \tfrac{1}{2}(\tau_1 - i\tau_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Just like spin, $\tau_\pm$ changes t_3 (i.e. t_0) without affecting total isospin t

$$\tau_\pm |t, t_0\rangle = \sqrt{t(t+1) - t_0(t_0 \pm 1)}\, |t, t_0 \pm 1\rangle$$

This formal expression, from raising and lowering operators, we'll need to understand beta decay rates of isobaric analog decays (beta decays really changing one proton to one neutron, almost.)

Isospin of quarks

$\tau_-|p\rangle \equiv \tau_-|uud\rangle = |n\rangle \equiv |udd\rangle$
similarly
$\tau_+|udd\rangle = |uud\rangle$

Also, t_0 just counts ('is a scalar quantity') so
$t_0|uud>= +1/2$ **and** $t_0|udd>= -1/2$

$\therefore$ u **has** $t_0 = +1/2$ **and** d **has** $t_0 = -1/2$**, a doublet ☺ in 'strong isospin'**

Wong p. 35 justifies why c, s, t, b quarks are not in isospin doublets. They have isospin 0. ☹
K mesons $u\bar{s}$ etc. have t=1/2, not t=3/2 (quoted as an experimental observation) and therefore s has isospin zero...

The SM doublets we see on charts all the time
$\binom{u}{d}$ $\binom{c}{s}$ $\binom{t}{b}$
are under $SU(2)_L$ weak interaction ('weak isospin')
Poll:

Isospin of antiparticles

under charge conjugation C, Wong uses 2nd-quantized operators to show that there's a phase to keep track of:

$|p\rangle \xrightarrow{C} (-1)^(t + t0)|\bar{p}\rangle = -|\bar{p}\rangle$

and similarly we may need

$|n\rangle \xrightarrow{C} (-1)^(1/2 - 1/2)|\bar{n}\rangle = +|\bar{n}\rangle$

$|u\rangle \xrightarrow{C} -|\bar{u}\rangle$

$|d\rangle \xrightarrow{C} +|\bar{d}\rangle$

Spin-0 pseudoscalar mesons from up, down quarks

π has spin 0, isospin 1; π^+ $t_3 = 1$; π^- $t_3 = -1$; π^0 $t_3 = 0$
Parity of π: (+ q parity)*(- $\bar{q}$ parity)* $(-1)^l$ for spatial wf; for $l = 0$ parity is -

- **To get electric charge -1, there's a unique combination:**

$|\pi^-\rangle = |\bar{u}d>$

Wong formally constructs the neutral π from the isospin raising operator:

$|\pi^0\rangle = \frac{1}{\mathrm{Norm}}\tau_+|\pi^-\rangle = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle - |d\bar{d}\rangle)$ **there's also a phase for the + one:**

$|\pi^+\rangle = -|u\bar{d}\rangle$ **(I would have to study these phases to use them ☹)**

- **masses: π^0 134.9766(6) MeV; π^+ 139.57018(35) MeV; π^+ can β decay to π^0.**

mean τ_e: π^0 8.4×10^{-17}s; π^+ is 2.6×10^{-8}s $\Leftarrow$ The $\pi^0 \rightarrow \gamma + \gamma$, much faster.

By inspection, there's another $t_3 = 0$ meson orthogonal to the π^0:

$|\eta\rangle = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle + |d\bar{d}\rangle)$ **The η has spin 0 and $t = 0$, and m_η = 550 MeV**

(I think these wf's are missing exchange terms. Is Wong just assuming we can do that trivially if we need it?)

Spin-1, negative parity, mesons from up, down quarks

ρ, ω mesons are also 'virtually' exchanged between nucleons– one of the ways to think about the strong interaction between nucleons.
one can couple the quark-antiquark together to spin 1 instead of 0.

ρ^- is $d\bar{u}$,
ρ^0 is $\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$,
ρ^+ is $u\bar{d}$
ρ^0 mass 775.3 MeV, mean lifetime 4.45 $\times 10^{-24}$s

ω is $\frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$,
mass 782.7 MeV, lifetime 7.8$\times 10^{-23}$s

Lifetimes shorter than π, because they decay by strong interaction

ρ^0 and ω do mix because of isospin symmetry breaking.
This changes the nucleon-nucleon interaction a little– e.g. there's a 5 keV extra mass difference between ^{3}He and ^{3}H that is unaccounted for by the Coulomb interaction (G. Miller arXiv:1810.05239)

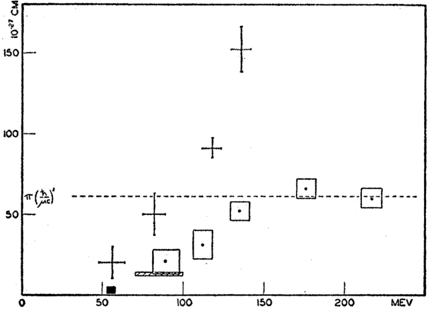


FIG. 1. Total cross sections of negative pions in hydrogen (sides of the rectangle represent the error) and positive pions in hydrogen (arms of the cross represent the error). The cross-hatched rectangle is the Columbia

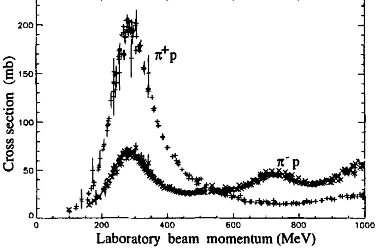


Figure 2-3: Total cross section of charged pions scattering off protons. The strong P_{33}-resonance in $\pi^+ + p$ reaction occurs in the $(J^\pi, T) = (\frac{3}{2}^+, \frac{3}{2})$ channel with $\ell = 1$. The $\pi^- + p$ cross section at the same energy is much smaller, as the

Δ resonance excitation and isospin

Anderson, Fermi, Long, Nagle Phys Rev 85 935 (1952)

$$\sigma(p + \pi^+ \to \Delta^{++}) \propto$$

$$|\langle t_p\, t_3(p)\; t_\pi t_3(\pi^+)|t_\Delta t_3(\Delta^{++})\rangle|^2 =$$

$$|\langle \tfrac{1}{2}\,\tfrac{1}{2}\;1 1|\tfrac{3}{2}\;\tfrac{3}{2}\rangle|^2 = 1$$

$$\sigma(p + \pi^- \to \Delta^0) \propto$$

$$|\langle t_p\, t_3(p)\; t_\pi t_3(\pi^-)|t_\Delta t_3(\Delta^0)\rangle|^2 =$$

$$|\langle \tfrac{1}{2}\,\tfrac{1}{2}\;1\; -1|\tfrac{3}{2}\; -\tfrac{1}{2}\rangle|^2 = 1/3 \text{ ☺}$$

Though Fermi et al. pointed out the exit channel branches matter, too (“K.A. Bruckner private comm.”)

unitarity limit $\sigma \propto \pi\bar{\lambda}^2$ independent of interaction strength (!!)

What else am I assuming?

Δ^{++} and need for 'color' quantum number

Using u and d, here are 4 spin 3/2, isospin $t = 3/2$ baryons:

Δ^{++} uuu

Δ^{+} uud, excited states of proton with spins all aligned

Δ^{0} udd, excited states of neutron with spins all aligned

Δ^{-} ddd

Consider antisymmetry. Δ's have noninteger spin, so are fermions, so wf's must be antisymmetric under exchange of particles.

Writing $\Delta^{++} = uuu \uparrow\uparrow\uparrow \psi_{\text{spatial}}$,

noting parity is known positive so ψ_{spatial} is symmetric

Δ^{++} is completely symmetric under exchange

We need another degree of freedom, the 'color' charge of QCD. A very powerful and fundamental statement, the need for a new quantum number to satisfy antisymmeterization i.e. Pauli exclusion principle

$$\Delta^{++} = uuu \uparrow\uparrow\uparrow \sqrt{\tfrac{1}{6}}(RGB - RBG + BRG - BGR + GBR - GRB)$$

This asymmetric wf also has zero color, a 'color singlet.'

Nucleon resonances PDG 2019:

Excited states of the nucleon, all from u and d quarks only.

Δ's are t=3/2, N's are t=1/2

$\Delta(1232)$ Γ_{FWHM} = "114 to 120 MeV"

$N(1440)$ Γ_{FWHM}= "250 to 450 MeV"
("Roper" resonance,
quarks perhaps in an excited spatial state with N=2? arXiv:1909.13732.v2)

Widths $\sim$ 10% of their mass: $\therefore$ unbound to strong interaction decays.

Does the p wf have admixture of N? Δ?

$$|p'\rangle = |p\rangle + \frac{|\Delta\rangle\langle p|H_{Coulomb}|\Delta\rangle}{\Delta E}$$

$$|p'\rangle \overset{?}{\approx} |p\rangle + 10^{-2\text{ to }-3}|\Delta\rangle$$

G-T quenching? (March) Poll

Particle	J^P	overall
$\Delta(1232)$	$3/2^+$	****
$\Delta(1600)$	$3/2^+$	****
$\Delta(1620)$	$1/2^-$	****
$\Delta(1700)$	$3/2^-$	****
$\Delta(1750)$	$1/2^+$	*
$\Delta(1900)$	$1/2^-$	***
$\Delta(1905)$	$5/2^+$	****
$\Delta(1910)$	$1/2^+$	****
$\Delta(1920)$	$3/2^+$	***
$\Delta(1930)$	$5/2^-$	***
$\Delta(1940)$	$3/2^-$	**
$\Delta(1950)$	$7/2^+$	****
$\Delta(2000)$	$5/2^+$	**
$\Delta(2150)$	$1/2^-$	*
$\Delta(2200)$	$7/2^-$	***
$\Delta(2300)$	$9/2^+$	**
$\Delta(2350)$	$5/2^-$	*
$\Delta(2390)$	$7/2^+$	*
$\Delta(2400)$	$9/2^-$	**
$\Delta(2420)$	$11/2^+$	****
$\Delta(2750)$	$13/2^-$	**
$\Delta(2950)$	$15/2^+$	**

**** Existence is certain.

N	$1/2^+$	****
$N(1440)$	$1/2^+$	****
$N(1520)$	$3/2^-$	****
$N(1535)$	$1/2^-$	****
$N(1650)$	$1/2^-$	****
$N(1675)$	$5/2^-$	****
$N(1680)$	$5/2^+$	****
$N(1700)$	$3/2^-$	***
$N(1710)$	$1/2^+$	****
$N(1720)$	$3/2^+$	****
$N(1860)$	$5/2^+$	**
$N(1875)$	$3/2^-$	***
$N(1880)$	$1/2^+$	***
$N(1895)$	$1/2^-$	****
$N(1900)$	$3/2^+$	****
$N(1990)$	$7/2^+$	**
$N(2000)$	$5/2^+$	**
$N(2040)$	$3/2^+$	*
$N(2060)$	$5/2^-$	***
$N(2100)$	$1/2^+$	***
$N(2120)$	$3/2^-$	***
$N(2190)$	$7/2^-$	****
$N(2220)$	$9/2^+$	****
$N(2250)$	$9/2^-$	****
$N(2300)$	$1/2^+$	**
$N(2570)$	$5/2^-$	**
$N(2600)$	$11/2^-$	***
$N(2700)$	$13/2^+$	**

Exchange symmetry Example/Review: coupling 2 spin-1/2 particles

Can couple two spin-1/2 particle to total S=1 or S=0:

$\frac{\uparrow\downarrow - \downarrow\uparrow}{\sqrt{2}}$	**S=0**	**Antisymmetric under 1↔2**
$\uparrow\uparrow, \quad \frac{\uparrow\downarrow + \downarrow\uparrow}{\sqrt{2}}, \quad \downarrow\downarrow$	**S=1** **m_S=1, 0, -1**	**Symmetric under 1↔2**

mantra: “the stretched state is always symmetric”

antisymmetric under exchange: nucleon wavefunction

One ansatz: $\psi = \psi_{\mathbf{flavour}}\psi_{\mathbf{spin}}\psi_{\mathbf{color}}$

Using the antisymmetric color singlet we used for the Δ^{++}, then $\psi_{\mathbf{flavour}}\psi_{\mathbf{spin}}$ must be symmetric under exchange

Both $\psi_{\mathbf{flavour}}$ and $\psi_{\mathbf{spin}}$ can be symmetric;

both can be antisymmetric; or,

one can break from the ansatz and have terms symmetric under exchange of the first pair and antisymmetric in the second pair.

Young tableaux i.e. Young diagrams are a good way to organize terms of mixed symmetry. I may use those later, because the mixed symmetry terms are needed for antisymmeterizing nuclear wf's

the proton wf:

1st 2 quarks S=0: $|\frac{1}{2},+\frac{1}{2}\rangle = \frac{1}{\sqrt{2}}\Big(|q(1)\uparrow\rangle|q(2)\downarrow\rangle - |q(1)\downarrow\rangle|q(2)\uparrow\rangle\Big)|q(3)\uparrow\rangle$

Assuming the same RGB antisymmetric wf for color we used for the Δ^{++}, here is the isospin symmetric wf multiplying the spin symmetric wf :

NOTE: It is possible to write a completely antisymmetric nucleon wavefunction without color! A problem constructs this, showing it gives wrong experimental μ.

Let us start by giving the first two quarks different flavors. Equation (2-44) becomes

$$|\tfrac{1}{2},+\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}}\Big(|u(1)\uparrow\rangle|d(2)\downarrow\rangle - |u(1)\downarrow\rangle|d(2)\uparrow\rangle\Big)|u(3)\uparrow\rangle$$

The combination of spin and flavor may be symmetrized in two stages. First we shall carry out the process only for the first two quarks and obtain

$$\begin{aligned}|\tfrac{1}{2},+\tfrac{1}{2}\rangle &= \tfrac{1}{2}\Big(|u(1)\uparrow\rangle|d(2)\downarrow\rangle - |u(1)\downarrow\rangle|d(2)\uparrow\rangle \\ &\quad +|d(1)\downarrow\rangle|u(2)\uparrow\rangle - |d(1)\uparrow\rangle|u(2)\downarrow\rangle\Big)|u(3)\uparrow\rangle \qquad (2\text{-}45)\end{aligned}$$

Next, we shall generate the others by applying permutations P_{31} and P_{32} on each of the four terms in Eq. (2-45). This gives us a total of 12 terms. On grouping identical terms together, we obtain the quark wave function for a proton with spin orientations of all the quarks indicated explicitly,

$$\begin{aligned}|p\rangle &= \frac{1}{\sqrt{18}}\Big\{2\Big(|u\uparrow u\uparrow d\downarrow\rangle + |u\uparrow d\downarrow u\uparrow\rangle + |d\downarrow u\uparrow u\uparrow\rangle\Big) \\ &\quad -\Big(|u\uparrow u\downarrow d\uparrow\rangle + |u\uparrow d\uparrow u\downarrow\rangle + |d\uparrow u\uparrow u\downarrow\rangle \\ &\quad +|u\downarrow u\uparrow d\uparrow\rangle + |u\downarrow d\uparrow u\uparrow\rangle + |d\uparrow u\downarrow u\uparrow\rangle\Big)\Big\} \qquad (2\text{-}46)\end{aligned}$$

from Wikipedia: $\mu_{\mathbf{proton}}$ and color

The anomalously large magnetic moment of the proton was discovered in 1933 by Otto Stern in Hamburg.[4][5] Stern won the Nobel Prize in 1943 for this discovery.[6]

By 1934 groups led by Stern, now in Pittsburgh, and I. I. Rabi in New York had independently measured the magnetic moments of the proton and deuteron.[7][8][9] While the measured values for these particles were only in rough agreement between the groups, the Rabi group confirmed the earlier Stern measurements that the magnetic moment for th proton was unexpectedly large.[10][11] Since a deuteron is

Both groups achieved 10% accuracy.
Stern was low by 1 σ.
Rabi was high by 2 σ: Nobel 1944 ☺

The value is much too big for a Dirac particle $\Rightarrow$ p has structure

Direct evidence for p substructure from e^- scattering at SLAC: 1967-73.

In one of the early successes of the Standard Model (SU(6) theory), in 1964 Mirza A. B. Beg, Benjamin W. Lee, and Abraham Pais theoretically calculated the ratio of proton to neutron magnetic moments to be −3/2, which agrees with th experimental value to within 3%.[20][21][22] The measured value for this ratio is −1.459 898 06(34).[23] A contradiction the quantum mechanical basis of this calculation with the Pauli exclusion principle led to the discovery of the color charge for quarks by Oscar W. Greenberg in 1964.[20]

Greenberg credits 'color as a gauge symmetry' to Nambu

20. ^ *a* *b* Greenberg, O. W. (2009), "Color charge degree of freedom in particle physics", *Compendium of Quantum Physics, ed. D. Greenberger, K. Hentschel and F. Weinert, (Springer-Verlag, Berlin Heidelberg P*: 109–111, arXiv:0805.0289, CiteSeerX 10.1.1.312.5798, doi:10.1007/978-3-540-70626-7_32, ISBN 978-3-540-70622-9, S2CID 17512393
21. ^ Beg, M.A.B.; Lee, B.W.; Pais, A. (1964). "SU(6) and electromagnetic interactions". *Physical Review Letters*. **13** (16): 514–517, erratum 650. Bibcode:1964PhRvL..13..514B. doi:10.1103/physrevlett.13.514.
22. ^ Sakita, B. (1964). "Electromagnetic properties of baryons in the supermultiplet scheme of elementary particles". *Physical Review Letters*. **13** (21): 643–646.

Magnetic moment of *p* in general

$\vec{\mu} = (g_l \vec{l} + g_s \vec{s}) \frac{q\hbar}{2mc}$ **Experimentally,** $\mu_p = (1 + 1.79)\frac{e\hbar}{2mc}$

The deviation from the pointlike Dirac value told people very early on the proton is not a pointlike Dirac particle

Since all particles have similar quantized angular momenta $\sim \hbar$, note the 1/*m* scaling. A problem guides you through the classical derivation of the orbital angular momentum part, to try to provide intuition.

$m_{\rm proton}$ **= 938 MeV**

$\therefore$ **If the quark mass were the ‘current’ mass in QCD, a few MeV, magnetic moments of *u*, *d* quarks would be much larger than for the nucleon**

Instead, we get close to the experimental answer iff we tune $m_{\rm quark} \approx m_{\rm nucleon}/3$, the “constituent” mass. People do calculate the ‘constituent mass’ from QCD binding energy...

Some textbooks are taking ratios of n and p magnetic moments, not comparing the absolute values:

magnetic moment of nucleon, quark model

$$\vec{\mu} = (g_l \vec{l} + g_s \vec{s}) \frac{q\hbar}{2mc}$$

$$\mu \overset{def}{=} \langle J, M = J | \vec{\mu} | J, M = J \rangle$$

$$\equiv \langle JJ | \mu_z | JJ \rangle$$

So μ operator is μ_z, does nothing to wf but project out the value.
So each term of these wf's remains orthogonal– you don't mix terms.
(A good test of calculated wf's in many physical systems.)
For nucleons, sum over the 3 constituents:

$$\mu_{\mathrm{p}} = \sum_{i=1}^{3} \mu_i$$

$$|p\rangle = \frac{1}{\sqrt{18}} \Big\{ 2\Big(|u\uparrow u\uparrow d\downarrow\rangle + |u\uparrow d\downarrow u\uparrow\rangle + |d\downarrow u\uparrow u\uparrow\rangle \Big)$$
$$-\Big(|u\uparrow u\downarrow d\uparrow\rangle + |u\uparrow d\uparrow u\downarrow\rangle + |d\uparrow u\uparrow u\downarrow\rangle$$
$$+ |u\downarrow u\uparrow d\uparrow\rangle + |u\downarrow d\uparrow u\uparrow\rangle + |d\uparrow u\downarrow u\uparrow\rangle \Big) \Big\}$$

Square the coefficients, notice the $\pm$ spin projections:

$$\frac{1}{18}\{4((\mu_u + \mu_u - \mu_d)$$
$$+(\mu_u - \mu_d + \mu_u) + (-\mu_d + \mu_u + \mu_u))$$
$$+\frac{1}{18}(0\mu_u + 6\mu_d)\ \}$$

$$\Rightarrow \mu_{\mathrm{p}} = \frac{4}{3}\mu_u - \frac{1}{3}\mu_d$$

for neutron, switch all u's to d's:

$$\Rightarrow \mu_{\mathrm{n}} = = \frac{4}{3}\mu_d - \frac{1}{3}\mu_u$$

Compare μ from quark model to experiment

$$\mu_{\mathrm{p}} = \frac{4}{3}\mu_u - \frac{1}{3}\mu_d$$
$$\mu_{\mathrm{n}} = \frac{4}{3}\mu_d - \frac{1}{3}\mu_u$$

- an individual magnetic moment

$$\mu_z^i = s_z^i \frac{q^i}{e} \frac{e\hbar}{m_q c}$$

scales with charge/mass

In 'constituent quark model' (?!)
Assume same mass (?!) so $\mu_u = -2\mu_d$ from the ratio of charges, then
μ_p = -3 μ_d and μ_n = 2 μ_d, and

- μ_p/μ_n = -1.5. Measured ratio is -1.46 ☺.

Further assuming $m_q \approx m_{\mathrm{nucleon}}/3$ (?!)

- μ_p = 3 μ_{nucleon} Measured value is 2.79 ☺
- μ_n = - 2 μ_{nucleon} Measured value is -1.91 ☺

This builds a consistent picture. Wong Table 2-4 shows similar results for other baryons including *s* quark.
But 'current' mass of the quarks in QCD is a few MeV, and $m_d > m_u$. If these classical ideas were the whole story, μ_u and μ_d should be 100x bigger.
This is part of the 'spin puzzle' where the spin of the nucleon also has components from the gluons (although a great deal of this is studying the spin structure at small distances by high momentum transfer). Also, people do work on calculating the 'constituent mass' by QCD binding energy.

μ of baryons

Table 2-4: Magnetic dipole moment of baryon octet.

Octet member	Quark content u	d	s	Best fit μ_N	Observed μ_N
p	$\frac{4}{3}$	$-\frac{1}{3}$	0	2.793	2.792847386(63)
n	$-\frac{1}{3}$	$\frac{4}{3}$	0	−1.913	−1.91304275(45)
Λ	0	0	1	−0.613	−0.613(4)
Σ^+	$\frac{4}{3}$	0	$-\frac{1}{3}$	2.674	2.458(10)
Σ^-	0	$\frac{4}{3}$	$-\frac{1}{3}$	−1.092	−1.160(25)
Ξ^0	$-\frac{1}{3}$	0	$\frac{4}{3}$	−1.435	−1.250(14)
Ξ^-	0	$-\frac{1}{3}$	$\frac{4}{3}$	−0.493	−0.6507(25)
$\Sigma^0 \to \Lambda$	$-\sqrt{\frac{1}{3}}$	$\sqrt{\frac{1}{3}}$	0	−1.630	−1.61(8)
Ω^-			3	−1.839	−2.02(5)
u	1			1.852	
d		1		−0.972	
s			1	−0.613	

- **Fit parameters μ_u, μ_d, μ_s to all the measured baryon magnetic moments, which are reproduced pretty well. Consistent with our nucleon-only numbers, but adding redundancy.**
- **Note μ_d and μ_u are pretty close to the factor of -2 from known charge ratio assuming same mass. Note μ_s is not close.**
- **These effective magnetic moments of the quarks while bound into hadrons seem to be guided by their 'constituent' masses. Martinelli PLB 116 434 (1982) lattice QCD for μ_p, μ_n**
It might be interesting to compare $\mu_\rho = 2.1 \pm 0.5 \frac{e}{2m_\rho}$ (BABAR 2015) in a quark model.

Electromagnetism alters μ_e in many-body systems

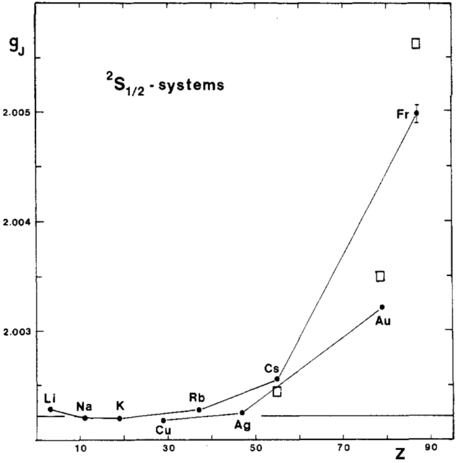


Fig. 2. Experimental g_J-values of the electronic ground states $ns\ ^2S_{1/2}$ in the alkali elements and in the noble metals. The result in francium is from the present work. A strong deviation from the horizontal line, representing the Landé value $g_J = 2.002319$, is observed with increasing atomic number. The theoretical results from Refs. [12, 13] are given by squares.

Ekström, Robertsson, Rosén, Phys Scripta Vol. 34 624 (1986)

- **The magnetic moment of the electron changes in high-Z atoms by 0.15%.**
- **Maybe it's unsurprising for μ_u, μ_d to be altered from 'free quark' values in a strongly-acting many-body system.**

Summary Phys 505 Lectures 2,3

- **Isospin is a good symmetry in the strong interactions, providing understanding of reactions and decays.**

- **Antisymmetry of fermion wf's (implementing Pauli exclusion) $\psi(1,2) = -1\,\psi(2,1)$ has many phenomenological consequences. To construct quark wf's for baryons that match experiment, one needs the quantum number 'color'**

- **Together, isospin symmetry and antisymmetry will have many consequences for nuclear structure that don't depend much on other details of the interaction.**

- **The magnetic moment of baryons is reproduced in the constituent quark model, a major success historically. This is likely trying to tell us that the constituent mass that comes as QCD binds the quarks into hadrons is the important driver of this physics, not the 'current' mass of the quarks in the QCD Lagrangian. Hence the 'spin puzzle' is being addressed at smaller scales probed at higher momentum transfer, eventually at the new EIC.**

Set 2 Prob 1: Classical scaling of μ with $\hbar$/mass

Work Wong Problem 2-6 for the classical nonrelativistic magnetic moment of a particle in terms of angular momentum. Go ahead and use Jackson's treatment (see next page). The goal is just to understand the simple scaling with (angular momentum)/mass. Given the same quantized angular momentum for anything, magnetic phenomena have vastly different scales for leptons and baryons, except in systems where their spin is coupled ("hyperfine interactions").

2-6. An electron is moving in a circular orbit. Show that the magnetic dipole moment generated by the orbital motion is given by the relation

$$\boldsymbol{\mu} = -\frac{e\hbar[c]}{2m_e c}\boldsymbol{\ell}$$

where $\boldsymbol{\ell}$ is the angular momentum in units of $\hbar$ and the factor $[c]$ converts the formula from cgs to SI units. Assume that the charge and mass of the electron are distributed uniformly along the orbit and ignore the contributions from the intrinsic magnetic dipole moment.

vector potential,

$$\mathbf{A}(\mathbf{x})=\frac{\mathbf{m}\times\mathbf{x}}{|\mathbf{x}|^3} \tag{5.55}$$

This is the lowest nonvanishing term in the expansion of $\mathbf{A}$ for a localized steady-state current distribution. The magnetic induction $\mathbf{B}$ outside the localized source can be calculated directly by evaluating the curl of (5.55):

$$\mathbf{B}(\mathbf{x})=\frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{m})-\mathbf{m}}{|\mathbf{x}|^3} \tag{5.56}$$

Here $\mathbf{n}$ is a unit vector in the direction $\mathbf{x}$. The magnetic induction (5.56) has exactly the form (4.13) of the field of a dipole. This is the generalization of the result found for the circular loop in the last section. Far away from *any* localized current distribution the magnetic induction is that of a magnetic dipole of dipole moment given by (5.54).

If the current is confined to a plane, but otherwise arbitrary, loop, the magnetic moment can be expressed in a simple form. If the current I flows in a closed circuit whose line element is $d\mathbf{l}$, (5.54) becomes

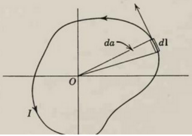
$$\mathbf{m}=\frac{I}{2c}\oint \mathbf{x}\times d\mathbf{l}$$

For a plane loop such as that in Fig. 5.7, the magnetic moment is perpendicular to the plane of the loop. Since $\frac{1}{2}|\mathbf{x}\times d\mathbf{l}|=da$, where da is the triangular element of the area defined by the two ends of $d\mathbf{l}$ and the origin, the loop integral gives the total area of the loop. Hence the magnetic moment has magnitude,

$$|\mathbf{m}|=\frac{I}{c}\times(\text{Area}) \tag{5.57}$$

regardless of the shape of the circuit.

If the current distribution is provided by a number of charged particles with charges q_i and masses M_i in motion with velocities $\mathbf{v}_i$, the magnetic moment can



be expressed in terms of the orbital angular momentum of the particles. The current density is

$$\mathbf{J}=\sum_i q_i\mathbf{v}_i\delta(\mathbf{x}-\mathbf{x}_i)$$

where $\mathbf{x}_i$ is the position of the ith particle. Then the magnetic moment (5.54) becomes

$$\mathbf{m}=\frac{1}{2c}\sum_i q_i(\mathbf{x}_i\times\mathbf{v}_i)$$

The vector product $(\mathbf{x}_i\times\mathbf{v}_i)$ is proportional to the ith particle's orbital angular momentum, $\mathbf{L}_i=M_i(\mathbf{x}_i\times\mathbf{v}_i)$. Thus the moment becomes

$$\mathbf{m}=\sum_i\frac{q_i}{2M_ic}\mathbf{L}_i \tag{5.58}$$

If all the particles in motion have the same charge to mass ratio $(q_i/M_i=e/M)$, the magnetic moment can be written in terms of the *total* orbital angular momentum $\mathbf{L}$:

$$\mathbf{m}=\frac{e}{2Mc}\sum_i\mathbf{L}_i=\frac{e}{2Mc}\mathbf{L} \tag{5.59}$$

This is the well-known classical connection between angular momentum and magnetic moment which holds for orbital motion even on the atomic scale. But this classical connection fails for the intrinsic moment of electrons and other elementary particles. For electrons, the intrinsic moment is slightly more than twice as large as implied by (5.59), with the spin angular momentum $\mathbf{S}$ replacing $\mathbf{L}$. Thus we speak of the electron having a g factor of 2(1.00116). The departure of the magnetic moment from its classical value has its origins in relativistic and quantum-mechanical effects which we cannot consider here.

Before leaving the topic of the fields of a localized current distribution, we consider the spherical volume integral of the magnetic induction $\mathbf{B}$. Just as in the electrostatic case discussed at the end of Section 4.1, there are two limits of interest, one in which the sphere of radius R contains all of the current and the other where the current is completely external to the spherical volume. The volume integral of $\mathbf{B}$ is

$$\int_{r<R}\mathbf{B}(\mathbf{x})\,d^3x=\int_{r<R}\nabla\times\mathbf{A}\,d^3x \tag{5.60}$$

The volume integral of the curl of $\mathbf{A}$ can be integrated to give a surface integral. Thus

Set 2 Probs 2 and 3: Isospin in reactions, decays

Problem 2, from Halzen and Martin:

EXERCISE 2.3 Use isospin invariance to show that the reaction cross sections σ must satisfy

$$\frac{\sigma(pp \to \pi^+ d)}{\sigma(np \to \pi^0 d)} = 2,$$

given that the deuteron d has isospin $I = 0$ and the π has isospin $I = 1$.

Hint You may assume that the reaction rate is

$$\sigma \sim |\text{amplitude}|^2 \sim \sum_I |\langle I', I_3' | A | I, I_3 \rangle|^2$$

where I and I' are the total isospin quantum numbers of the initial and final states, respectively, and $I = I'$ and $I_3 = I_3'$.

Problem 3:

Similarly compute the ratio of the decay rates for:

$\Delta_0 \to p + \pi^-$

$\Delta_0 \to n + \pi^0$

Mention briefly what you are assuming. Accelerator-based ν oscillation students need to know how accurate such branch calculations are, since those two decay channels look very different in a detector.

There are two tables of C-G's on the next 2 pages, from Wong Appendix A-1 and from a Particle Data Group classic table

34. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND *d* FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2	Coefficients		
m_1	m_2			
.	.			
.	.			
.	.			

1/2×1/2

m_1	m_2		
	J	1	
	M	+1	
+1/2	+1/2	1	
	J	1	0
	M	0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2
	J	1	
	M	−1	
−1/2	−1/2	1	

1×1/2

m_1	m_2		
	J	3/2	
	M	+3/2	
+1	+1/2	1	
	J	3/2	1/2
	M	+1/2	+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3
	J	3/2	1/2
	M	−1/2	−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3
	J	3/2	
	M	−3/2	
−1	−1/2	1	

2×1

m_1	m_2			
	J	3		
	M	+3		
+2	+1	1		
	J	3	2	
	M	+2	+2	
+2	0	1/3	2/3	
+1	+1	2/3	−1/3	
	J	3	2	1
	M	+1	+1	+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10
	J	3	2	1
	M	0	0	0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10
	J	3	2	1
	M	−1	−1	−1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5
	J	3	2	
	M	−2	−2	
−1	−1	2/3	1/3	
−2	0	1/3	−2/3	
	J	3		
	M	−3		
−2	−1	1		

1×1

m_1	m_2			
	J	2		
	M	+2		
+1	+1	1		
	J	2	1	
	M	+1	+1	
+1	0	1/2	1/2	
0	+1	1/2	−1/2	
	J	2	1	0
	M	0	0	0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3
	J	2	1	
	M	−1	−1	
0	−1	1/2	1/2	
−1	0	1/2	−1/2	
	J	2		
	M	−2		
−1	−1	1		

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

3/2×1

m_1	m_2			
	J	5/2		
	M	+5/2		
+3/2	+1	1		
	J	5/2	3/2	
	M	+3/2	+3/2	
+3/2	0	2/5	3/5	
+1/2	+1	3/5	−2/5	
	J	5/2	3/2	1/2
	M	+1/2	+1/2	+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6
	J	5/2	3/2	1/2
	M	−1/2	−1/2	−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2
	J	5/2	3/2	
	M	−3/2	−3/2	
−1/2	−1	3/5	2/5	
−3/2	0	2/5	−3/5	
	J	5/2		
	M	−5/2		
−3/2	−1	1		

2×1/2

m_1	m_2		
	J	5/2	
	M	+5/2	
+2	+1/2	1	
	J	5/2	3/2
	M	+3/2	+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5
	J	5/2	3/2
	M	+1/2	+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5
	J	5/2	3/2
	M	−1/2	−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5
	J	5/2	3/2
	M	−3/2	−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5
	J	5/2	
	M	−5/2	
−2	−1/2	1	

3/2×1/2

m_1	m_2		
	J	2	
	M	+2	
+3/2	+1/2	1	
	J	2	1
	M	+1	+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4
	J	2	1
	M	0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2
	J	2	1
	M	−1	−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4
	J	2	
	M	−2	
−3/2	−1/2	1	

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

Table A-1: Some useful Clebsch-Gordan coefficients.

$\langle \frac{1}{2} m_s \ell m_\ell | j m \rangle$

j	$m_s = +\frac{1}{2}$	$m_s = -\frac{1}{2}$
$\ell + \frac{1}{2}$	$\sqrt{\frac{\ell + \frac{1}{2} + m}{2\ell + 1}}$	$\sqrt{\frac{\ell + \frac{1}{2} - m}{2\ell + 1}}$
$\ell - \frac{1}{2}$	$\sqrt{\frac{\ell + \frac{1}{2} - m}{2\ell + 1}}$	$-\sqrt{\frac{\ell + \frac{1}{2} + m}{2\ell + 1}}$

$\langle 1 m_s \ell m_\ell | j m \rangle$

j	$m_s = +1$	$m_s = 0$	$m_s = -1$
$\ell + 1$	$\sqrt{\frac{(\ell + m)(\ell + m + 1)}{2(\ell + 1)(2\ell + 1)}}$	$\sqrt{\frac{(\ell - m + 1)(\ell + m + 1)}{(\ell + 1)(2\ell + 1)}}$	$\sqrt{\frac{(\ell - m)(\ell - m + 1)}{2(\ell + 1)(2\ell + 1)}}$
ℓ	$\sqrt{\frac{(\ell + m)(\ell - m + 1)}{2\ell(\ell + 1)}}$	$\frac{-m}{\sqrt{\ell(\ell + 1)}}$	$-\sqrt{\frac{(\ell - m)(\ell + m + 1)}{2\ell(\ell + 1)}}$
$\ell - 1$	$\sqrt{\frac{(\ell - m)(\ell - m + 1)}{2\ell(2\ell + 1)}}$	$-\sqrt{\frac{(\ell - m)(\ell + m)}{\ell(2\ell + 1)}}$	$\sqrt{\frac{(\ell + m + 1)(\ell + m)}{2\ell(2\ell + 1)}}$

$$\begin{pmatrix} j & 0 & j' \\ -m & 0 & m' \end{pmatrix} = \langle j m j' -m' | 00 \rangle = \frac{(-1)^{j-m}}{\sqrt{2j+1}} \delta_{jj'} \delta mm' \qquad \langle j m 0 0 | j' m' \rangle = \delta_{jj'} \delta_{mm'}$$

$$\begin{pmatrix} j & 1 & j \\ -m & 0 & m \end{pmatrix} = (-1)^{j-m} \frac{m}{\sqrt{j(j+1)(2j+1)}}$$

$$\begin{pmatrix} j & 2 & j \\ -m & 0 & m \end{pmatrix} = (-1)^{j-m} \frac{3m^2 - j(j+1)}{\sqrt{(2j-1)j(j+1)(2j+1)(2j+3)}}$$

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} = \begin{cases} (-1)^g \sqrt{\frac{(2g - 2j_1)!(2g - 2j_2)!(2g - 2j_3)!}{(2g+1)!}} \frac{g!}{(g - j_1)!(g - j_2)!(g - j_3)!} & \text{if } 2g = \text{ even} \\ 0 & \text{if } 2g = \text{ odd} \end{cases}$$

where $2g = j_1 + j_2 + j_3$

Prob 4: Halzen+Martin: An antisymmeterized *p* wf without color

EXERCISE 2.18 The spin-flavor wavefunctions of the ground-state baryons are symmetric, and color was invoked to recover the required antisymmetric character. You should notice, however, and some people did, that we can construct a totally antisymmetric proton wavefunction, for example,

$$|p\uparrow\rangle = \sqrt{\tfrac{1}{2}}\,[p_A\,\chi(M_S) - p_S\,\chi(M_A)],$$

and forget about color! Write this function in an explicit form, comparable to (2.71). Obtain $|n\uparrow\rangle$, and hence show that

$$\frac{\mu_n}{\mu_p} = -2.$$

So this option is ruled out by experiment. In fact, glancing at your derivation, you will notice that μ_p is negative. It is measured to be positive. Long live color.

Halzen and Martin go through the finite group theory needed to justify these functions. Here I just assume and check them:

Eq 2.71 is symmetric in spin, ud

$$|p\uparrow\rangle = \sqrt{\tfrac{1}{2}}\,(p_S\chi(M_S) + p_A\chi(M_A))$$

$$|p\uparrow\rangle = \sqrt{\tfrac{1}{18}}\,[\mathrm{uud}(\uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow - 2\uparrow\uparrow\downarrow) + \mathrm{udu}(\uparrow\uparrow\downarrow + \downarrow\uparrow\uparrow - 2\uparrow\downarrow\uparrow)$$
$$+\,\mathrm{duu}(\uparrow\downarrow\uparrow + \uparrow\uparrow\downarrow - 2\downarrow\uparrow\uparrow)]$$

M_S and ***M_A*** spin 'up' functions symmetric, antisymmetric in 1st 2 pair of spins:

$$\chi(M_S) = \sqrt{\tfrac{1}{6}}\,(\uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow - 2\uparrow\uparrow\downarrow)$$
$$\chi(M_A) = \sqrt{\tfrac{1}{2}}\,(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow).$$

p_S, ***p_A*** are handily symmetric, antisymmetric in the 1st 2 quarks:

$$p_S = \sqrt{\tfrac{1}{6}}\,[(\mathrm{ud} + \mathrm{du})\mathrm{u} - 2\mathrm{uud}]$$

$$p_A = \sqrt{\tfrac{1}{2}}\,(\mathrm{ud} - \mathrm{du})\mathrm{u}$$