HW4 Phys505 on Lecture 6-7. Due Fri Feb 10 9:30 am JB solutions at end

Levinson's theorem, np scattering

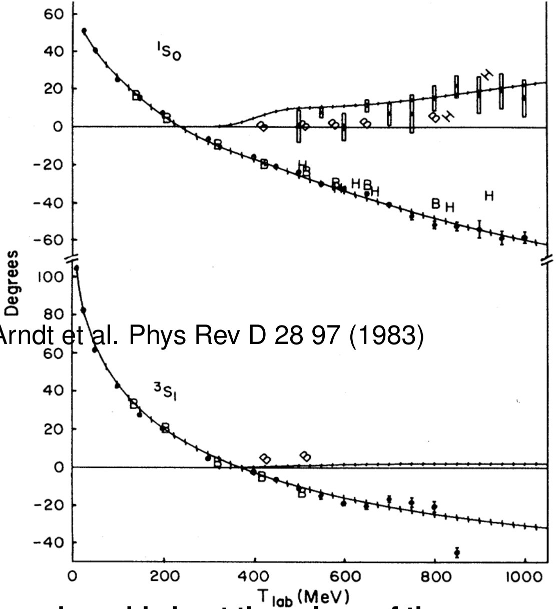
1) There are challenges deducing info about a potential from the phase shifts. One tool is "Levinson's Theorem," :

- **The difference in the ℓ-wave phase shift of a scattered wave at zero energy, $\phi_\ell(0)$, and infinite energy, $\phi_\ell(\infty)$, for a spherically symmetric potential $V(r)$, is related to the number of bound states n_ℓ with angular momentum l of the potential by:**

$$\phi_\ell(0) - \phi_\ell(\infty) = n_\ell \pi$$

(Wellner, American Journal of Physics 32, 787 (1964); https://doi.org/10.1119/1.1969857 .)

1a) Using only data plotted here for the phase shifts, determine the effective number of bound states (a fraction) and estimate its uncertainty. Is this a meaningful constraint on the number of bound states of the deuteron?

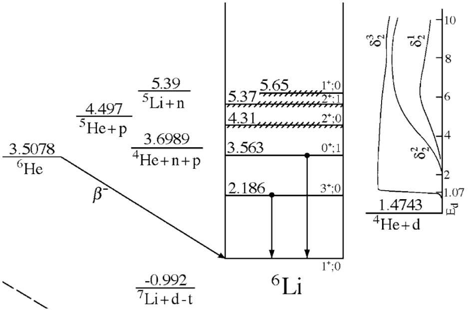


Arndt et al. Phys Rev D 28 97 (1983)

1b) Instead, assume n_0=1, which it is. What can be said about the values of the phase shift extrapolated to 0 and infinite energy?

Decay of excited 3.563 MeV $J^\pi=0^+$; T=1 state of 6Li

This state, the isobaric analog of the ^{6}He ground state, is energetically allowed to decay to d + α (and not be emitting neutron or proton).
Yet it has only been observed to γ decay.
2a) Would the α decay obey isospin symmetry? (Isospin-'forbidden' particle decay is routinely observed– the decay rate is much slower than if isospin is allowed.)
2b) Assuming total angular momentum is conserved (smooth rotations are good!), what orbital angular momentum would be needed for the final *d* $+ \alpha$ system? Would this decay preserve parity symmetry? (Note: the weak interaction between nucleons can break parity, and has both isoscalar and isovector components.)



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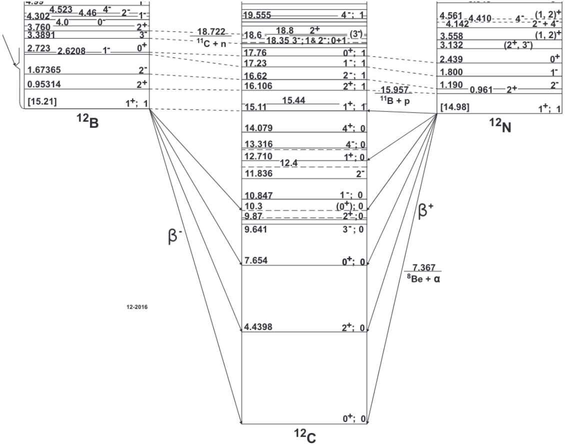
Isospin mixing in ^{12}C

See pages 20-21 of L07_IsospinInNuclei_JB_2023_v2.pdf:

3a) Assuming the expression involving $\tan\theta$ (similar to 1st-order perturbation theory but comes from diagonalizing the 2x2), what size matrix element of $H_{\rm Coul}$ is needed to produce $\theta \sim 40^{\rm o}$ between ^{8}Be 16.6 and 16.9 MeV states? JB thinks Wong Eq. 4-54 is a small-angle approximation

3b) Scaling this matrix element by Z, guesstimate the size of the Coulomb matrix element in ^{12}C between the $J^\pi = 1^+$ 12.7 MeV (mostly T=0) and $J^\pi = 1^+$ 15.11 MeV (mostly T=1) states.



(This is another case of analog anti-analog mixing.) What prediction results for the amount of T=0 admixture in the 15.11 MeV state's wf?

3c) Assuming the Coulomb operator is spherically symmetric, is there a Coulomb-produced admixture between the 12.7 MeV $J^\pi = 1^+$ and the 16.106 $J^\pi = 2^+$ states?

Signs Consider the lowest-order chiral EFT Lagrangian in coordinate space:

$$V^{(\mathrm{LO})}(\boldsymbol{r}) = (C_S + C_T \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2)\delta(\mathbf{r}) - \left(\frac{g_A}{2f_\pi}\right)^2 (\boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2)(\boldsymbol{\sigma}_1 \cdot \nabla_1)(\boldsymbol{\sigma}_2 \cdot \nabla_2)\frac{e^{-m_\pi r}}{4\pi r}, \quad (2.79)$$

from e.g. Obertelli and Sagawa

C_S and C_T are determined by:

4a) Fitting to data (e.g. nucleon-nucleon scattering)?

4b) Underlying symmetries of QCD? **(Don't spend much time on this– it may or may not be very subtle)**

4c) Lattice QCD computation?

Consider an alternate universe with the signs of C_S and C_T both flipped.

4d) State the physics change to the nucleon-nucleon interaction

4e) List two possible changes to the resulting physics and/or universe

From Wikipedia (unchecked, sorry), the β function of QCD assuming SU(3) is :

$$\beta(\alpha_S) = -(11 - \frac{Nc}{6} - \frac{2n_q}{3})\frac{\alpha_S^2}{2\pi}$$

A positive definite coupling out front seems reasonable, but there's a typo concerning g^3 and α_S^2: caveat emptor

where $Nc = 3$ colors of gluons, and n_q is the number of flavors of quarks, in our case 6 so far. (A more general expression for SU(Nc) won and lost the Nobel.)

The sign of β is famously negative, implying asymptotic freedom.

4f) Assuming both asymptotic freedom and SU(3) are well-established, what is the maximum n_q possible in our observed universe?

Energies of *d* and *d*-like states

OPEP, generated by Problem 4's Lagrangian, includes the "tensor force" that is said to bind the deuteron. By inspection this term scales with $\vec{\tau}_1 \cdot \vec{\tau}_2$.

5a) What is $\vec{\tau}_1 \cdot \vec{\tau}_2$ for the two deuteron states? (Wong Ch. 3; JB p. 13 L06 NN Int: just state the answer.)

However, fully written-out OPEP:

e.g. N. Jelley, Fund Nucl Phys Cambridge free .pdf at publisher

$$V_{\mathrm{P}} = g_s^2 \left(\tfrac{1}{3} \boldsymbol{\sigma}_A \cdot \boldsymbol{\sigma}_B + S_{AB} \left[\tfrac{1}{3} + \frac{1}{\mu r} + \frac{1}{(\mu r)^2} \right] \right) \boldsymbol{\tau}_A \cdot \boldsymbol{\tau}_B \frac{\mu^2 e^{-\mu r}}{r}$$

where $S_{AB} = 3(\boldsymbol{\sigma}_A \cdot \mathbf{r})(\boldsymbol{\sigma}_B \cdot \mathbf{r})/r^2 - \boldsymbol{\sigma}_A \cdot \boldsymbol{\sigma}_B$.

shows dependence on $\vec{\sigma}_1 \cdot \vec{\sigma}_2$ intertwined in the "tensor force."

5b) What is $\vec{\sigma}_1 \cdot \vec{\sigma}_2$ for the two deuteron states? JB hopes this is obvious from 5a

JB: so these 2 simple quantities flip with the *d*'s states: the full answer for *d* binding requires less handwaving and more radial integration.

Note from Problem 2's diagram: the g.s. of ^{6}Li has T=0, $J \neq 0$, $\pi+$ like the deuteron.

That *J*=1 is the same as in the *d* is coincidental: the same permutation symmetry means these $p_{1/2}$ orbitals add their total angular momenta to maximum. Similarly, the 1st excited J=3 state adds two $p_{3/2}$ orbitals to max total angular momentum.

Also, the 2nd excited state of ^{6}Li has the same J^π, *T* as the excited (unbound) state of the *d*, and same permutation symmetry (symmetric $T = 1$, antisymmetric $J = 0$.) Such 'deuteron-symmetry' states exist for odd-N = odd-Z to quite high A.

5c) What is the lightest odd-N = odd-Z nucleus where these deuteron-like states invert their energy order? I.e. has ground state J^π=0$^+$ g.s. ?

(mouse over the wall chart at www.nndc.bnl.gov to get g.s. properties once you pick a nucleus and zoom in.)

What drives this inversion (spin-orbit, size, 'isospin dependence of pairing'...) is beyond JB so far

solutions HW4

1a) Ground state is spin 1, so we want the $^{2s+1}L_J = {}^3S_1$ channel. Extrapolating from the data here gives greater than 145 degrees, perhaps $\pm$ 20, $\Rightarrow n_0 \approx 0.8 \pm 0.1$. Consistent with 1 bound state.
(Note we have no $J = 0$ bound state. The $^1S_0$ state has phase shift difference 120 degrees, considerably less than 180, though the extrapolation to zero energy needs more data than shown here.)

1b) The phase shift at infinite energy looks to be no less than -40. The phase shift at lower energy needs to be 140 degrees– it seems possible that could be so.

These comparisons suggest Levinson's theorem is an elegant math physics result, but when n_ℓ is 0 or 1 it is difficult to interpret.

Addendum: I interpret the $^1S_0$ channel as relating to the $J^\pi; T = 0^+; 1$ resonance unbound at 80 keV. The phase shift difference has a substantial deficit from π. Although that seems intuitively reasonable that a slightly unbound resonance could have such a phase shift, one would have to look carefully at the math physics proof to check if that is a rigorous conclusion.

Solutions HW4 continued

2a) ^{6}Li 3.563 MeV state has $J^\pi ; T = 0^+; 1$. Since deuteron and α have g.s. with $T = 0$, the decay has to take place by an isospin-breaking interaction, or the g.s. has to have an admixture of $T = 1$ from an isospin-breaking interaction.
(One could in principle emit d or α in an excited state with $T = 1$, though these are all unbound to nucleon emission. p.s. People have measured temperatures in compound nuclei by looking at the ratio of d^*/d emission and using a Boltzmann relation including the extra energy needed to create d^*, cleverly indentifying d^* by correlation of n,p emission (though the extraction of temperature left out a critical isospin coupling).

2b) Such a decay would also need orbital angular momentum L=1 between the d and α, so the outgoing state has parity -1. So this decay would also break parity symmetry.

Published calculations using the parameterized weak nucleon-nucleon interaction– its isovector component– predict about 10^{-4} for this branch.
A measurement would be an interesting test of our understanding, though I'm aware only of upper limits on the branch.

HW4 Problem 3 page 1

Thanks to students in 2021 for trying to use Wong Eq. 4-54 and the 2021 TA and one student for figuring out the discrepancy and bringing to my attention the very different answer. Both my notes and Wong in principle lay out a 2-level mixing problem the same way, with the same notation.

The answer should be tan(theta), not sin(theta), for (matrix element)/(energy difference), and this matters at 10-20%. However, I suspect Wong is using another small-angle approximation incorrectly, as Eq. 4-54 is failing badly at large angles, while Eq. 4-54 is consistent with my relations at small angles. I don't recommend trying any harder to prove or verify Eq. 4-54– I respect those who have tried, while I find experimental literature using tan(theta) = (matrix element)/(energy difference).

Extra info: I've amended the lecture notes to include a step, for ^{8}Be experimental decay rates Γ:

$$\frac{\Gamma_\alpha(16.9)}{\Gamma_\alpha(16.6)} = \tan^2(\theta) = 0.69$$

which leads to the angle of 40 degrees I wrote down. This relation assumes α 'penetrabilities' are the same. Note that would not work for the very different energies in ^{12}C, where a full technical review of α and M1 γ decay (isovector + isoscalar) is in Adelberger et al. PRC 15 484 (1977) Section III.

HW4 Problem 3

3a) L07, p. 22/23 for ^{8}Be: $\sin\theta = 0.64 \Rightarrow$
$\langle T = 1|H_{\text{Coul}}|T = 0\rangle = \sin\theta \times (16.922 - 16.626) = 0.19$ MeV

3b) scaling by Z, 6/4, gives estimate 0.27 MeV for ^{12}C states.
(This is naive, ignoring details of the wf's in the analog and antianalog states; more detailed consideration in the McDonald and Adelberger paper has ^{12}C's matrix element about 10% larger than ^{8}Be)

1st-order perturbation theory $\Rightarrow$ $\sqrt{Admixture}$ = $\langle T = 1|H_{\text{Coul}}|T = 0\rangle/\Delta E$ =
0.27 MeV/2.41 MeV = 0.11,
so Admixture $\sim$ 0.012, much smaller than ^{8}Be simply because two-level mixing amplitude scales with 1/(energy splitting)
if you kept track of the sign of the energy denominator, that is indeed important for some predicted observables

3c) No. To mix states, one needs nonzero $\langle J = 1|H_{\text{Coulomb}}|J = 2\rangle$.
The Coulomb interaction inside a uniform charged sphere (then $H_{\text{Coulomb}} \sim r^2$) is spherically symmetric, so it can't change the angular momentum, so this matrix element vanishes by inspection.

The delta function is part of the hard-core short-range repulsion of the nucleon-nucleon interaction. (Obertelli and Sagawa note that higher-order terms in the chiral EFT nucleon-nucleon expansion also contribute.)
4a) yes, experiments show hard-core repulsion
4b) probably not (though see 4c)). It seems unrelated to symmetries of QCD reflected in the chiral EFT Lagrangian itself, and to asymptotic freedom.
4c) Yes, it is considered a major success of the attempts to calculate the nucleon-nucleon interaction from lattice QCD that the hard core repulsion is reproduced at all (the plot is in O&S and the lecture notes). What feature of QCD drives the hard core repulsion is another question. Lattice QCD (and certain observables in the 1/Nc expansion) is a unique way to provide answers from QCD in nonperturbative regimes, but different effects must be hard to isolate in such a nonlinear problem.

One point of problem 4 is to ask whether some prominent features of the nucleon-nucleon interaction are driven from QCD. Despite 4c):
4d) The hard-core repulsion would flip sign, becoming a hard-core attraction
4e) Answers include: Nuclei would have smaller radii, with higher binding energies. Scattering σ at higher energy would change (though lowest-order Coulomb scattering does not depend on sign...). Stable nuclei could have larger imbalance of p and n. Energetics of nuclear astrophysics and element abundances would change. Would A=5, A=8 would have stable isotopes?– see lectures soon on permutation symmetry.

4f) $n_q = 15$ is ok, n_q=16 would flip the sign and alter many effects of QCD at higher energy. There are very few things that depend on the number of particle families.

People tell me there are other expressions, even in Wikipedia, sorry.

p.s. t'Hooft is credited with working out the full expression for SU(N), a year earlier, but not publishing it. Maybe he didn't notice the sign?

The goal here was to see if simple scaling answers the question of which ***d*** state should have lower energy. JB has not gained much insight into this interesting question from this problem, sorry.

Pasting in notes: in Wong Ch 2 $\boldsymbol{t}(\boldsymbol{t}+\mathbf{1}) = (\frac{1}{2})(\frac{3}{2}) = \mathbf{3/4}$

(Pauli matrices are constructed from 'unity', so there are factors of 1/2)

$\vec{\tau}^2 = 4t(t+1) = 3$

so $\vec{\tau_1} \cdot \vec{\tau_2}$ is an isospin scalar, and a two-body operator, that distinguishes T=0 and T=1 states: $\langle T|\vec{\tau_1} \cdot \vec{\tau_2}|T\rangle$ = -3 for T=0, 1 for T=1

5a) i.e. -3 for the T=0 *d* ground state, +1 for the T=1 *d* excited unbound resonance

$\vec{\tau_1} \cdot \vec{\tau_2}$ scaling of this potential helps determine which ***d*** state has lower energy

5b) $\vec{\sigma_1} \cdot \vec{\sigma_2}$ is just reversed, +1 for the J=S=1 *d* ground state, -3 for the excited *d* resonance.

To answer the question of whether either ***d*** state is bound requires evaluating $\vec{\sigma_i} \cdot \vec{r}$, a spatial wavefunction, and radial integrals with this N-N potential term.

5c) ^{34}Cl. Such nuclei break the rule of thumb that ground states have T=T_3

p.s. We may cover that E1 photon multipole is isovector. The giant dipole (E1) resonance excitation of an even-even $N = Z$ $T = 0$ ground state will group a number of T=1 1^- states in the continuum. There are other specific nuclear structures that group states of same *T*.