Set 2 Prob 1: Classical scaling of μ with $\hbar$/mass

Work Wong Problem 2-6 for the classical nonrelativistic magnetic moment of a particle in terms of angular momentum. Go ahead and use Jackson's treatment (see next page). The goal is just to understand the simple scaling with (angular momentum)/mass. Given the same quantized angular momentum for anything, magnetic phenomena have vastly different scales for leptons and baryons, except in systems where their spin is coupled ("hyperfine interactions").

2-6. An electron is moving in a circular orbit. Show that the magnetic dipole moment generated by the orbital motion is given by the relation

$$\boldsymbol{\mu} = -\frac{e\hbar[c]}{2m_e c}\boldsymbol{\ell}$$

where $\boldsymbol{\ell}$ is the angular momentum in units of $\hbar$ and the factor $[c]$ converts the formula from cgs to SI units. Assume that the charge and mass of the electron are distributed uniformly along the orbit and ignore the contributions from the intrinsic magnetic dipole moment.

vector potential,

$$\mathbf{A}(\mathbf{x})=\frac{\mathbf{m}\times\mathbf{x}}{|\mathbf{x}|^3} \tag{5.55}$$

This is the lowest nonvanishing term in the expansion of $\mathbf{A}$ for a localized steady-state current distribution. The magnetic induction $\mathbf{B}$ outside the localized source can be calculated directly by evaluating the curl of (5.55):

$$\mathbf{B}(\mathbf{x})=\frac{3\mathbf{n}(\mathbf{n}\cdot\mathbf{m})-\mathbf{m}}{|\mathbf{x}|^3} \tag{5.56}$$

Here $\mathbf{n}$ is a unit vector in the direction $\mathbf{x}$. The magnetic induction (5.56) has exactly the form (4.13) of the field of a dipole. This is the generalization of the result found for the circular loop in the last section. Far away from *any* localized current distribution the magnetic induction is that of a magnetic dipole of dipole moment given by (5.54).

If the current is confined to a plane, but otherwise arbitrary, loop, the magnetic moment can be expressed in a simple form. If the current I flows in a closed circuit whose line element is $d\mathbf{l}$, (5.54) becomes

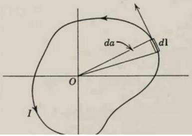
$$\mathbf{m}=\frac{I}{2c}\oint \mathbf{x}\times d\mathbf{l}$$

For a plane loop such as that in Fig. 5.7, the magnetic moment is perpendicular to the plane of the loop. Since $\frac{1}{2}|\mathbf{x}\times d\mathbf{l}|=da$, where da is the triangular element of the area defined by the two ends of $d\mathbf{l}$ and the origin, the loop integral gives the total area of the loop. Hence the magnetic moment has magnitude,

$$|\mathbf{m}|=\frac{I}{c}\times(\text{Area}) \tag{5.57}$$

regardless of the shape of the circuit.

If the current distribution is provided by a number of charged particles with charges q_i and masses M_i in motion with velocities $\mathbf{v}_i$, the magnetic moment can



be expressed in terms of the orbital angular momentum of the particles. The current density is

$$\mathbf{J}=\sum_i q_i\mathbf{v}_i\delta(\mathbf{x}-\mathbf{x}_i)$$

where $\mathbf{x}_i$ is the position of the ith particle. Then the magnetic moment (5.54) becomes

$$\mathbf{m}=\frac{1}{2c}\sum_i q_i(\mathbf{x}_i\times\mathbf{v}_i)$$

The vector product $(\mathbf{x}_i\times\mathbf{v}_i)$ is proportional to the ith particle's orbital angular momentum, $\mathbf{L}_i=M_i(\mathbf{x}_i\times\mathbf{v}_i)$. Thus the moment becomes

$$\mathbf{m}=\sum_i\frac{q_i}{2M_ic}\mathbf{L}_i \tag{5.58}$$

If all the particles in motion have the same charge to mass ratio ($q_i/M_i=e/M$), the magnetic moment can be written in terms of the *total* orbital angular momentum $\mathbf{L}$:

$$\mathbf{m}=\frac{e}{2Mc}\sum_i\mathbf{L}_i=\frac{e}{2Mc}\mathbf{L} \tag{5.59}$$

This is the well-known classical connection between angular momentum and magnetic moment which holds for orbital motion even on the atomic scale. But this classical connection fails for the intrinsic moment of electrons and other elementary particles. For electrons, the intrinsic moment is slightly more than twice as large as implied by (5.59), with the spin angular momentum $\mathbf{S}$ replacing $\mathbf{L}$. Thus we speak of the electron having a g factor of 2(1.00116). The departure of the magnetic moment from its classical value has its origins in relativistic and quantum-mechanical effects which we cannot consider here.

Before leaving the topic of the fields of a localized current distribution, we consider the spherical volume integral of the magnetic induction $\mathbf{B}$. Just as in the electrostatic case discussed at the end of Section 4.1, there are two limits of interest, one in which the sphere of radius R contains all of the current and the other where the current is completely external to the spherical volume. The volume integral of $\mathbf{B}$ is

$$\int_{r<R}\mathbf{B}(\mathbf{x})\,d^3x=\int_{r<R}\nabla\times\mathbf{A}\,d^3x \tag{5.60}$$

The volume integral of the curl of $\mathbf{A}$ can be integrated to give a surface integral. Thus

Set 2 Probs 2 and 3: Isospin in reactions, decays

Problem 2, from Halzen and Martin:

EXERCISE 2.3 Use isospin invariance to show that the reaction cross sections σ must satisfy

$$\frac{\sigma(pp \to \pi^+ d)}{\sigma(np \to \pi^0 d)} = 2,$$

given that the deuteron d has isospin $I = 0$ and the π has isospin $I = 1$.

Hint You may assume that the reaction rate is

$$\sigma \sim |\text{amplitude}|^2 \sim \sum_I \left|\langle I', I_3' | A | I, I_3 \rangle\right|^2$$

where I and I' are the total isospin quantum numbers of the initial and final states, respectively, and $I = I'$ and $I_3 = I_3'$.

Problem 3:

Similarly compute the ratio of the decay rates for:

$\Delta_0 \to p + \pi^-$

$\Delta_0 \to n + \pi^0$

Mention briefly what you are assuming. Accelerator-based ν oscillation students need to know how accurate such branch calculations are, since those two decay channels look very different in a detector.

There are two tables of C-G's on the next 2 pages, from Wong Appendix A-1 and from a Particle Data Group classic table

34. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2	Coefficients		
m_1	m_2			
.	.			
.	.			

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

1/2×1/2

m_1	m_2	$J=1$	$J=0$
M = +1			
+1/2	+1/2	1	
M = 0			
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2
M = −1			
−1/2	−1/2	1	

1×1/2

m_1	m_2	$J=3/2$	$J=1/2$
M = +3/2			
+1	+1/2	1	
M = +1/2			
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3
M = −1/2			
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3
M = −3/2			
−1	−1/2	1	

2×1

m_1	m_2	$J=3$	$J=2$	$J=1$
M = +3				
+2	+1	1		
M = +2				
+2	0	1/3	2/3	
+1	+1	2/3	−1/3	
M = +1				
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10
M = 0				
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10
M = −1				
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5
M = −2				
−1	−1	2/3	1/3	
−2	0	1/3	−2/3	
M = −3				
−2	−1	1		

1×1

m_1	m_2	$J=2$	$J=1$	$J=0$
M = +2				
+1	+1	1		
M = +1				
+1	0	1/2	1/2	
0	+1	1/2	−1/2	
M = 0				
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3
M = −1				
0	−1	1/2	1/2	
−1	0	1/2	−1/2	
M = −2				
−1	−1	1		

3/2×1

m_1	m_2	$J=5/2$	$J=3/2$	$J=1/2$
M = +5/2				
+3/2	+1	1		
M = +3/2				
+3/2	0	2/5	3/5	
+1/2	+1	3/5	−2/5	
M = +1/2				
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6
M = −1/2				
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2
M = −3/2				
−1/2	−1	3/5	2/5	
−3/2	0	2/5	−3/5	
M = −5/2				
−3/2	−1	1		

2×1/2

m_1	m_2	$J=5/2$	$J=3/2$
M = +5/2			
+2	+1/2	1	
M = +3/2			
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5
M = +1/2			
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5
M = −1/2			
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5
M = −3/2			
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5
M = −5/2			
−2	−1/2	1	

3/2×1/2

m_1	m_2	$J=2$	$J=1$
M = +2			
+3/2	+1/2	1	
M = +1			
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4
M = 0			
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2
M = −1			
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4
M = −2			
−3/2	−1/2	1	

Table A-1: Some useful Clebsch-Gordan coefficients.

$\langle \frac{1}{2} m_s \ell m_\ell | j m \rangle$

j	$m_s = +\frac{1}{2}$	$m_s = -\frac{1}{2}$
$\ell + \frac{1}{2}$	$\sqrt{\frac{\ell + \frac{1}{2} + m}{2\ell + 1}}$	$\sqrt{\frac{\ell + \frac{1}{2} - m}{2\ell + 1}}$
$\ell - \frac{1}{2}$	$\sqrt{\frac{\ell + \frac{1}{2} - m}{2\ell + 1}}$	$-\sqrt{\frac{\ell + \frac{1}{2} + m}{2\ell + 1}}$

$\langle 1 m_s \ell m_\ell | j m \rangle$

j	$m_s = +1$	$m_s = 0$	$m_s = -1$
$\ell + 1$	$\sqrt{\frac{(\ell+m)(\ell+m+1)}{2(\ell+1)(2\ell+1)}}$	$\sqrt{\frac{(\ell-m+1)(\ell+m+1)}{(\ell+1)(2\ell+1)}}$	$\sqrt{\frac{(\ell-m)(\ell-m+1)}{2(\ell+1)(2\ell+1)}}$
ℓ	$\sqrt{\frac{(\ell+m)(\ell-m+1)}{2\ell(\ell+1)}}$	$\frac{-m}{\sqrt{\ell(\ell+1)}}$	$-\sqrt{\frac{(\ell-m)(\ell+m+1)}{2\ell(\ell+1)}}$
$\ell - 1$	$\sqrt{\frac{(\ell-m)(\ell-m+1)}{2\ell(2\ell+1)}}$	$-\sqrt{\frac{(\ell-m)(\ell+m)}{\ell(2\ell+1)}}$	$\sqrt{\frac{(\ell+m+1)(\ell+m)}{2\ell(2\ell+1)}}$

$$\begin{pmatrix} j & 0 & j' \\ -m & 0 & m' \end{pmatrix} = \langle j m j' \, -m' | 00 \rangle = \frac{(-1)^{j-m}}{\sqrt{2j+1}} \delta_{jj'} \delta mm' \qquad \langle j m 0 0 | j' m' \rangle = \delta_{jj'} \delta_{mm'}$$

$$\begin{pmatrix} j & 1 & j \\ -m & 0 & m \end{pmatrix} = (-1)^{j-m} \frac{m}{\sqrt{j(j+1)(2j+1)}}$$

$$\begin{pmatrix} j & 2 & j \\ -m & 0 & m \end{pmatrix} = (-1)^{j-m} \frac{3m^2 - j(j+1)}{\sqrt{(2j-1)j(j+1)(2j+1)(2j+3)}}$$

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} = \begin{cases} (-1)^g \sqrt{\frac{(2g-2j_1)!(2g-2j_2)!(2g-2j_3)!}{(2g+1)!}} \frac{g!}{(g-j_1)!(g-j_2)!(g-j_3)!} & \text{if } 2g = \text{ even} \\ 0 & \text{if } 2g = \text{ odd} \end{cases}$$

where $2g = j_1 + j_2 + j_3$

Prob 4: Halzen+Martin: An antisymmeterized *p* wf without color

EXERCISE 2.18 The spin-flavor wavefunctions of the ground-state baryons are symmetric, and color was invoked to recover the required antisymmetric character. You should notice, however, and some people did, that we can construct a totally antisymmetric proton wavefunction, for example,

$$|p\uparrow\rangle = \sqrt{\tfrac{1}{2}}\left[p_A\chi(M_S) - p_S\chi(M_A)\right],$$

and forget about color! Write this function in an explicit form, comparable to (2.71). Obtain $|n\uparrow\rangle$, and hence show that

$$\frac{\mu_n}{\mu_p} = -2.$$

So this option is ruled out by experiment. In fact, glancing at your derivation, you will notice that μ_p is negative. It is measured to be positive. Long live color.

Halzen and Martin go through the finite group theory needed to justify these functions. Here I just assume and check them:

Eq 2.71 is symmetric in spin, ud

$$|p\uparrow\rangle = \sqrt{\tfrac{1}{2}}\left(p_S\chi(M_S) + p_A\chi(M_A)\right)$$

$$|p\uparrow\rangle = \sqrt{\tfrac{1}{18}}\left[\mathrm{uud}(\uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow - 2\uparrow\uparrow\downarrow) + \mathrm{udu}(\uparrow\uparrow\downarrow + \downarrow\uparrow\uparrow - 2\uparrow\downarrow\uparrow)\right.$$
$$\left. + \mathrm{duu}(\uparrow\downarrow\uparrow + \uparrow\uparrow\downarrow - 2\downarrow\uparrow\uparrow)\right]$$

***M*$_S$** and ***M*$_A$** spin 'up' functions symmetric, antisymmetric in 1st 2 pair of spins:

$$\chi(M_S) = \sqrt{\tfrac{1}{6}}(\uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow - 2\uparrow\uparrow\downarrow)$$
$$\chi(M_A) = \sqrt{\tfrac{1}{2}}(\uparrow\downarrow\uparrow - \downarrow\uparrow\uparrow).$$

***p*$_S$**, ***p*$_A$** are handily symmetric, antisymmetric in the 1st 2 quarks:

$$p_S = \sqrt{\tfrac{1}{6}}\left[(\mathrm{ud} + \mathrm{du})\mathrm{u} - 2\mathrm{uud}\right]$$

$$p_A = \sqrt{\tfrac{1}{2}}(\mathrm{ud} - \mathrm{du})\mathrm{u}$$

5) Compute the Magnetic moment μ of the $\Delta^0(1232)$ resonance:

A wavefunction for the Δ^0 with isospin $t = 3/2$, $t_0 = t_3 = -1/2$, and intrinsic spin pointed fully up $S_z = S = 3/2$ is (Wong page 45):

$$|\psi_{\mathbf{spin}}\rangle\, |\psi_{\mathbf{isospin}}\rangle = |\uparrow\uparrow\uparrow\rangle\, |ddu + dud + udd\rangle/\sqrt{3}$$

a) This part is symmetric under permutation. State the permutation symmetry assumed of the spatial and color parts for this to be true (no proofs nor calculation needed).

b) Calculate the magnetic moment in terms of quark magnetic moments $\mu_{\mathbf{up}}$ and $\mu_{\mathbf{down}}$

c) Given known electric charges, and assuming mass $m_{\mathbf{up}} = m_{\mathbf{down}}$, eliminate $\mu_{\mathbf{up}}$. (The simplicity of this answer for μ of the Δ^0 is preserved by a published lattice gauge QCD calculation.) Compare to the neutron.

d) List 2 examples of extra physics that could alter this μ computation for the Δ^0 (no proofs nor calculation needed).

6) Parity and spontaneous symmetry breaking

Muñoz-Vega et al. Am. J. Physic **80** 891 (2012) (posted in Files on Canvas) addresses spontaneous symmetry breakdown SSB in non-relativistic QM. (It should be obvious their notation for what we called a symmetry operator/constant $\boldsymbol{Q}$ is $\boldsymbol{U}$.)

They construct states of equal energy in such a system related by the parity operator $\mathcal{P}$, their example of $\boldsymbol{Q}$. We will see in L12 and L21 SSB applied to deformation in nuclei including static octupole deformation.

5a) 5b) Fill in the simple steps proving eqs. 14, 16, 17: I have circled what I want on the 3rd page.