Astronomy Comprehensive Exam, Spring 2025

Session 1

May 09 2025

Note: if you are in the PhD in physics program, stop! This is the astronomy version of the exam. Please ask the proctor for the version appropriate for your program instead.

Do not write your name on your exam papers. Instead, write your student number on each page. This will allow us to grade the exams anonymously. We'll match your name with your student number after we finish grading.

This portion of the exam has 4 questions. Answer ***any three*** of the four. Do not submit answers to more than 3 questions—if you do, only the first 3 of the questions you attempt will be graded. If you attempt a question and then decide you don't want to it count, clearly cross it out and write "don't grade".

You have 2 hours 15 minutes to complete 3 questions.

You are allowed to use two $8.5'' \times 11''$ formula sheets (both sides), and a handheld, non-graphing calculator.

Here is a possibly useful table of physical constants and formulas:

atomic mass unit	1 amu	1.661×10^{-27} kg
Avogadro's constant	N_A	6.02×10^{23}
Bohr radius of hydrogen atom	a_0	5.3×10^{-11} m
Boltzmann's constant	k_B	1.38×10^{-23} J/K
charge of an electron	e	1.6×10^{-19} C
astronomical unit	1 au	1.5×10^{11} m
electron volt	1 eV	1.6×10^{-19} J
mass of an electron	m_e	0.5110 MeV/c^2
mass of hydrogen atom	m_H	1.674×10^{-27} kg
mass of a neutron	m_n	1.675×10^{-27} kg = 939.5654 MeV/c^2
mass of a proton	m_p	1.673×10^{-27} kg = 938.2721 MeV/c^2
mass of the moon	M_{moon}	7×10^{22} kg
mass of the sun	M_{sun}	2×10^{30} kg
Newton's gravitational constant	G	6.7×10^{-11} N m^2 kg^{-2}
permittivity of free space	ϵ_0	8.9×10^{-12} C^2 N^{-1}/m^2
permeability of free space	μ_0	$4\pi \times 10^{-7}$ N/A^2
Planck's constant	h	6.6×10^{-34} J·s
radius of the earth	$R_\oplus$	6.4×10^{6} m
radius of the sun	$R_\odot$	7×10^{8} m
speed of light	c	3.0×10^{8} m/s
Stefan-Boltzmann constant	σ	5.67×10^{-8} W m^{-2} K^{-4}
Stirling's approximation	$N!$	$e^{-N} N^N \sqrt{2\pi N}$
Laplacian in spherical coordinates	$\nabla^2\psi =$	$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$
Laplacian in cylindrical coordinates	$\nabla^2\psi =$	$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\psi}{\partial R}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$

1. A small planet orbits a star of mass M. A cloud of dust with a uniform mass density ρ surrounds the star and extends past the planet's orbit.
   (a) Calculate the orbital period of the planet as a function of the planet's distance from the star, assuming a circular orbit.
   (b) Now consider an orbit that is slightly non-circular, but has the same angular momentum, for which the planet's distance from the star varies as $R(t) = R_0[1+\zeta(t)]$. Show that the orbit precesses. (Hint: calculate the effective potential $V(r)$ of this system and consider small oscillations around its minimum.)
2. Radiation by a white dwarf derives its energy from the thermal energy of its ions, since the electrons form a degenerate Fermi gas and cannot lose energy easily. The luminosity of a white dwarf can be shown to relate to its mass and core temperature according to $L = AMT^{3.5}$, where M is in units of solar masses and $A = 0.2$ W/K$^{3.5}$.
   (a) Estimate the total heat capacity of a one solar mass white dwarf composed primarily of carbon.
   (b) How long would it take this white dwarf to cool from a core temperature of 10^7 K degrees to a temperature of 10^6 K degrees?
   (c) What would the surface temperature of the white dwarf be when its core temperature is 10^6 K? Assume that the radius of the white dwarf is $R = 1.0R_\oplus$.
3. As a simple model of a supernova, consider a sphere of initial radius 5×10^8 cm and mass $1.4M_\odot$, initially heated to 10^{10} K. Assuming constant density, total ionization, $Z/A = 0.5$, homologous expansion ($v \propto r$), and opacity due to electron scattering, calculate the radius the expanding sphere would have when it first became optically thin. (use $\kappa_e = 0.2$ cm^{2}g^{-1} throughout) If the expansion were adiabatic as well as homologous what would be the temperature of the radiation at this point (assume that radiation entropy is separately conserved)?
4. Assuming that dark energy consists of a simple cosmological constant, estimate Ω_m, Ω_Λ, and the energy density in the CMB
   (a) today
   (b) at $z = 2$
   (c) right after the time of recombination

Astronomy Comprehensive Exam, Spring 2025

Session 2

May 09 2025

Note: if you are in the PhD in physics program, stop! This is the astronomy version of the exam. Please ask the proctor for the version appropriate for your program instead.

Do not write your name on your exam papers. Instead, write your student number on each page. This will allow us to grade the exams anonymously. We'll match your name with your student number after we finish grading.

This portion of the exam has 4 questions. Answer ***any three*** of the four. Do not submit answers to more than 3 questions—if you do, only the first 3 of the questions you attempt will be graded. If you attempt a question and then decide you don't want to it count, clearly cross it out and write "don't grade".

You have 2 hours 15 minutes to complete 3 questions.

You are allowed to use two $8.5'' \times 11''$ formula sheets (both sides), and a handheld, non-graphing calculator.

Here is a possibly useful table of physical constants and formulas:

atomic mass unit	1 amu	1.661×10^{-27} kg
Avogadro's constant	N_A	6.02×10^{23}
Bohr radius of hydrogen atom	a_0	5.3×10^{-11} m
Boltzmann's constant	k_B	1.38×10^{-23} J/K
charge of an electron	e	1.6×10^{-19} C
astronomical unit	1 au	1.5×10^{11} m
electron volt	1 eV	1.6×10^{-19} J
mass of an electron	m_e	0.5110 MeV/c^2
mass of hydrogen atom	m_H	1.674×10^{-27} kg
mass of a neutron	m_n	1.675×10^{-27} kg = 939.5654 MeV/c^2
mass of a proton	m_p	1.673×10^{-27} kg = 938.2721 MeV/c^2
mass of the moon	M_{moon}	7×10^{22} kg
mass of the sun	M_{sun}	2×10^{30} kg
Newton's gravitational constant	G	6.7×10^{-11} N m^2 kg^{-2}
permittivity of free space	ϵ_0	8.9×10^{-12} C^2 N^{-1}/m^2
permeability of free space	μ_0	$4\pi \times 10^{-7}$ N/A^2
Planck's constant	h	6.6×10^{-34} J·s
radius of the earth	$R_\oplus$	6.4×10^6 m
radius of the sun	$R_\odot$	7×10^8 m
speed of light	c	3.0×10^8 m/s
Stefan-Boltzmann constant	σ	5.67×10^{-8} W m^{-2} K^{-4}
Stirling's approximation	$N!$	$e^{-N} N^N \sqrt{2\pi N}$
Laplacian in spherical coordinates	$\nabla^2\psi =$	$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$
Laplacian in cylindrical coordinates	$\nabla^2\psi =$	$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\psi}{\partial R}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$

5. Suppose that a differentiated Solar System body of radius R has a density distribution that can be approximated as a linear function of the distance from the centre r. Let ρ_1 be the central density and ρ_0 the density at the surface. In other words,

$$\rho(r) = \rho_1 + (\rho_0 - \rho_1)\,\frac{r}{R}.$$

Determine the internal pressure profile $P(r)$ within the body. Assume there is no substantial atmosphere around the body.

6. The NFW profile is commonly used to represent the spherically symmetric *dark matter* distribution of a galaxy and has density distribution

$$\rho(r) = \frac{\rho_0 R_s}{r(1 + r/R_S)^2}$$

where ρ_0 and R_s are normalization and scale parameters. Note that ρ_0 is *not* the density at $r = 0$.

   (a) Show that the corresponding potential is

   $$\Phi(r) = -\frac{4\pi G \rho_0 R_s^3}{r}\,\ln\left(1 + \frac{r}{R_s}\right)$$

   Note: The following integral may be helpful:

   $$\int \frac{x}{(1+x)^2} = \ln(1+x) - \frac{x}{1+x} + C$$

   (b) Compute the mass in dark matter from $r = R_{min}$ to $r = R_{max}$, and diagnose whether it diverges in the limits $R_{min} \to 0$ and/or $R_{max} \to \infty$. In any limits in which it diverges, explaining a reasonable astronomically-motivated cut-off for the radius in question.

   (c) A recent study based on the rotation-curve data derived from the Gaia spacecraft mission suggests that $R_s \simeq 12$ kpc. Accept this value and use the common estimate for the local dark-matter density in the solar neighbourhood of 0.01 $M_\odot$ pc^{-3} to estimate ρ_0. If you are unable to estimate ρ_0, then use $\rho_0 = 0.002 M_\odot$ pc^{-3} in part (d).

   (d) Using your cut-offs from (b), compute an estimated mass of the Milky Way's dark matter halo and comare it to estimated mass in stars of $6 \times 10^{10} M_\odot$ (Licquia and Newman 2015).

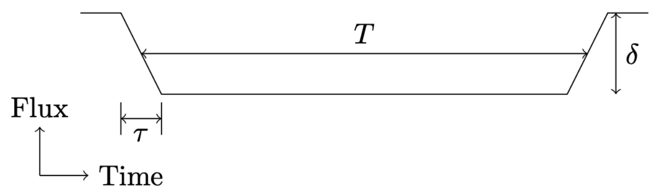
7. Suppose that you have two tiny pieces of metal. They are so small, in fact, that the internal energy in each piece is contained within five harmonic oscillators. The two pieces of metal are brought into thermal contact and reach equilibrium with ten units of energy ($\hbar\omega$ of the oscillators) in total. You can measure how much energy is in each piece of metal, but not which oscillators contain the energy.

   (a) How many macrostates are there?

   (b) How many total microstates are there?

   (c) Assuming that the system is ergodic, what would the chance be of seeing all of the energy in one piece of metal (either one)?

8. A planet transits a star of known mass M and radius R. You measure the transit depth δ, the duration of the onset of the transit τ, and the duration of the transit itself T measured from the midpoint of the onset to the midpoint of the ending as shown below.



Derive expressions for the orbital speed v of the planet, the period P and inclination i of the orbit. You should assume that the orbit is circular, the mass of the planet is much less than that of the star, the star and planet are spherical, and that there is no limb darkening.