January 2008 Comprehensive Exam for Advancement to Candidacy
Day 1: January 7, 2008

Exam ID stickers: The proctor will provide everyone with two stickers with serials numbers written on them. Make sure the numbers on both stickers match. Stick one on the back of your student ID. Stick the other on the front of your exam book. *Do not lose your sticker—this is the only identifier linking your exam booklet to you!* You'll need to show your sticker to claim credit for a passing mark. If you use extra exam booklets, write your serial number on the extra exam books by hand.

Today's portion of the exam has 9 questions. Answer *any six* of the nine. Do not submit answers to more than 6 questions—if you do, only the first 6 of the questions you attempt will be graded. If you attempt a question and then decide you don't want to it count, clearly cross it out and write "don't grade".

You have 4 hours to complete 6 questions.

You are allowed to use one $8'' \times 11''$ handwritten formula sheet (both sides), and a handheld calculator.

Here is a possibly useful table of physical constants:

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1. (a) Assuming that the atmosphere has a constant temperature (independent of the altitude), estimate the height at which the density of the atmosphere is half that at sea level. You can approximate that air is made entirely of diatomic nitrogen (molecular weight = 28 amu).

(b) Suppose now you have a hot air balloon of radius 10 meters. Given that the balloon and payload weigh a total of 200 kg when the balloon is empty, derive an expression for the height the balloon will fly as a function of the temperature of the gas inside the balloon. *Hint: the air pressure at sea level is 101,000 Pascal (N/m^2).*

(c) A more accurate calculation would account for the fact that the atmospheric temperature does vary with altitude. So drop the assumption that the temperature is constant, and instead suppose that the pressure and temperature of air at different altitudes are related to each other by an adiabatic expansion. (In other words, if you took a volume of air at sea level and move that air up, the density will decrease because the volume of the air expands adiabatically.) Show that for an adiabatic expansion

$$\frac{dP}{P} = \frac{\gamma}{\gamma - 1} \frac{dT}{T}$$

where $\gamma = 1.4$ is the adiabatic exponent for diatomic nitrogen.

Use this to then calculate the change in temperature of the atmosphere in degrees K if you go up 1 km, assuming that P and T are related according to this expression.

2. (a) Derive the orbital period of a planet orbiting a star of mass M in a circular orbit of radius R.

(b) The orbits of several stars have been measured orbiting the massive black hole at the centre of our Galaxy. One of them has an orbital period of 15 years, and the orbital radius is 0.12 arcseconds (as seen from Earth). (Note: one degree = 3600 arcseconds.) Take the distance to the centre of our Galaxy to be 8,000 pc (1 pc = 206,265 AU, 1 AU = 1.5×10^8 km). Compute the mass of the black hole. Express your answer in units of the Sun's mass (mass Sun 2×10^{30} kg). Assume that Newton's Law of Gravity is applicable for orbits sufficiently far from a black hole, and that the orbiting star satisfies this condition.

3. A particle of mass m and charge q is trapped into a harmonic potential $V(x) = kx^2/2$ (assume that only one-dimensional motion along the x-axis is possible). Besides this, an electric field $\vec{E} = E\hat{x}$ is also turned on.

(a) Assuming classical motion, find the position $x(t)$ of the particle, if $x(0) = v(0) = 0$.

(b) Assuming quantum motion, what is the ground state energy of this particle if we turn off the field ($E = 0$)? What is the expectation value of the dipole moment of the particle, $d \equiv qx$, in the ground state for $E = 0$?

(c) Find the exact ground-state energy of the particle in the case that the field strength $E > 0$. Now what is the expectation value of the dipole moment of the particle, $d \equiv qx$, in the ground state?

(d) Assume that we have N such systems, in equilibrium at a temperature T (there are no interactions between the different particles). Find how the dipole moment and its variance $\langle (d - \langle d \rangle)^2 \rangle$ vary with temperature. Sketch both curves as a function of T.

4. (a) Refer to the figure below. When the switch is opened, the light bulb continues to glow brightly for some time. Why? Where does the energy come from? Derive an expression for a quantity proportional to the optical power irradiated by the bulb as a function of time after the switch is opened. (In this circuit R_2 represents the internal resistance of the light bulb, which you can assume to be constant.)

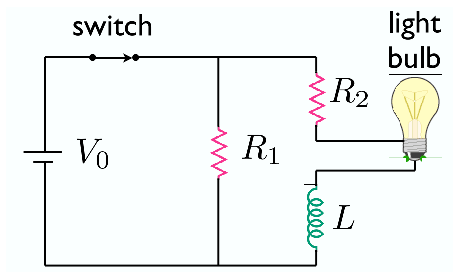

Figure 1:

(b) Refer to the figure below. The initial situation is indicated in the sketch: the capacitor C_1 is charged up with charges $+Q$ and $-Q$; the capacitor C_2 has no charge on either plate; the resistance R and the inductance L are doing nothing while the switches are open. At time $t = 0$, the switches are closed. Derive an expression for the voltage across C_1 as a function of time.

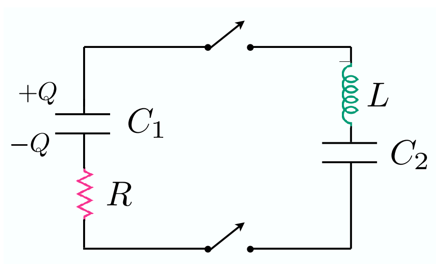

Figure 2:

5. (a) A bicycle travels in a straight line at moderate speed. In order to initiate a right turn, the rider initially must swing the handlebars to the *left* (counterclockwise), with the immediate consequence that the bike will lean into the right turn. Explain why turning the handlebars to the left initiates a right turn. (Hints: draw the velocity vectors for the vehicle just before and just after turning the wheel counterclockwise. Draw the free body diagram for the vehicle just after turning the wheel counterclockwise.)

(b) Once initiated, it is relatively easy to adjust the speed and angle of inclination to execute a turn with a given radius of curvature. Assuming that the centre of gravity of the bike and rider (considered a single rigid body of mass M) is a distance L above the ground when the bike is vertical, and that the mass of each wheel is m, and the wheel radius is r, show that the relationship between the steady state lean angle θ, the speed v, and the turning radius R is

$$\tan\theta = \frac{v^2}{R}\frac{2mr + ML}{MLg}$$

Discuss why this "makes sense" physically. (Hint: look at the equation in the limit that the mass of the rider and bike, and the height of the centre of gravity, are much larger than the mass and radius of the wheel respectively.)

6. Black holes have a radius of $R = GM/c^2$, and a thermodynamic entropy given by

$$S = \frac{8\pi^2 GM^2 k}{hc}$$

(a) Derive the temperature of a black hole as a function of its mass

(b) Black holes can lose mass by radiating energy (Hawking radiation, which you can assume is thermal). How long does it take a black hole of mass M to evaporate completely? Explain your reasoning and derive a formula for how long it takes.

7. Magneto-tactic bacteria are small rod-shaped organisms about 1 μm in length and 0.2 μm in diameter that can orient themselves by sensing the direction of the Earth's magnetic field. They have average densities about 10% higher than the density of the water they live in.

(a) Do an order of magnitude calculation that demonstrates that these bacteria cannot reliably tell "up" from "down" using the gravitational field gradient across their body. (Hint: consider the effects of thermal motion).

(b) Each bacterium in fact contain small crystals of magnetite Fe_3O_4, a ferri-magnetic material that acts like a compass needle inside the bacterium. If the Earth's magnetic field has a strength of 0.5 Gauss in Vancouver, estimate the approximate size of the bacterium's internal magnetic moment needed in order for it to be able to reliably sense the direction of this field.

(c) The bacteria use the magnetic field direction primarily to distinguish up from down. Treating the Earth's magnetic field as a dipole field aligned with the Earth's axis of rotation, calculate the angle between the field direction in Vancouver (latitude=49°N) and the down direction. (Note: pretend the Earth's magnetic poles are aligned with the rotational poles, and ignore the fact that in reality the magnetic north pole is displaced from the true north pole.)

8. A rocket of mass m_0 is propelled by a giant monochromatic laser mounted on the back of the rocket. The laser emits a beam with a power of P watts and a frequency of f_0, both measured in the rest frame of the rocket. When the beam is turned on, the rocket is driven in the opposite direction by the recoil.

(a) At $t = 0$ the laser is turned on, with the rocket initially at rest in the Earth's reference frame. Calculate the instantaneous acceleration of the rocket.

(b) If the rocket is moving at velocity v, what is the instantaneous beam power as measured in the Earth's reference frame?

(c) The laser is kept going until the speed of the rocket reaches $v = 0.9c$. What is the rest mass of the rocket at this point?

9. At time $t = 0$ the wavefunction of the hydrogen atom is

$$\psi(\vec{r}, 0) = \frac{1}{\sqrt{10}}(2\psi_{100} + \psi_{210} + \sqrt{2}\psi_{211} + \sqrt{3}\psi_{21-1})$$

where the subscripts are values of the quantum numbers n, l, m (let's here ignore spin and radiative transitions).
(a) What is the expectation value for the energy of this system? (Hint: the ground state energy of hydrogen is -13.6 eV.)

(b) What is the probability of finding the system with $l = 1, m = +1$ as a function of time?

(c) What is the probability of finding the electron within 10^{-10} cm of the proton (at time $t = 0$)? A good approximate result is acceptable here. Note that the radial wavefunctions $R_{nl}(r)$ for a hydrogen atom are:

$$\begin{aligned} R_{10}(r) &\propto \exp(-r/a_0) \\ R_{21}(r) &\propto r \, \exp(-r/2a_0) \end{aligned}$$

where $a_0 = 5.3 \times 10^{-11}$ m is the Bohr radius.

(d) How does this wave function evolve in time, i.e. what is $\psi(r, t)$?

January 2008 Comprehensive Exam for Advancement to Candidacy
Day 2: January 8, 2008

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1. Two long, hollow, and coaxial conducting cylinders, with radii a and b, are lowered into a tub of fluid with dielectric constant κ. A voltage V is applied between the two cylinders. The fluid is observed to rise up some height h into the volume between the cylinders. Calculate h.

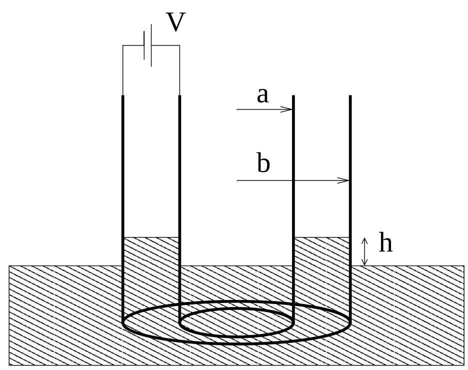

Figure 3:

2.

A quantum mechanical potential in one dimension is defined by:

$$V(x) = \begin{cases} 20 & x < 0 \\ 0 & 0 \le x \le 1 \\ 40 & x > 1 \end{cases}$$

In this problem set $\hbar = 1$, and the mass $m = 1/2$. There are two bound states $E_1 \approx 5.06$ and $E_2 \approx 18.25$, and a continuum of unbound states such as E_3 and E_4, as indicated on the graph.

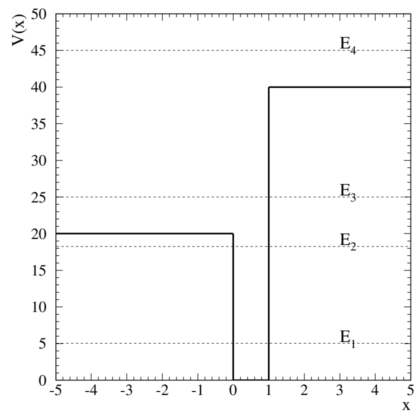

Figure 4:

(a) Qualitatively sketch the spatial part of the wavefunction $\psi(x)$ for each of the four energies shown in the diagram.

(b) Consider instead a potential given by

$$V(x) = \begin{cases} 0 & x \le 0 \\ V_0 & x > 0 \end{cases}$$

A particle with mass $m = 1/2$ and energy $E > V_0$ approaches the potential from the left. What is the probability that the particle is reflected back towards $x = -\infty$? (You can again set $\hbar = 1$.)

3.

An atomic interferometer is oriented in the vertical plane in a gravitational field of strength g. Atoms of mass m are injected at the bottom left with velocity v. The beam is split by a beam-splitter, with one half deflected straight up and the other half continuing horizontally. The split beams bounce off "atomic mirrors" (upper left and bottom right) and pass through a second beam-splitter in the upper right. A detector (indicated by the star) counts the number of atoms arriving. As the separation x between the two vertical arms of the interferometer is varied, the count rate changes as shown in the bottom graph. Calculate the strength of the gravitational field g as a function of m, y, D, and v.

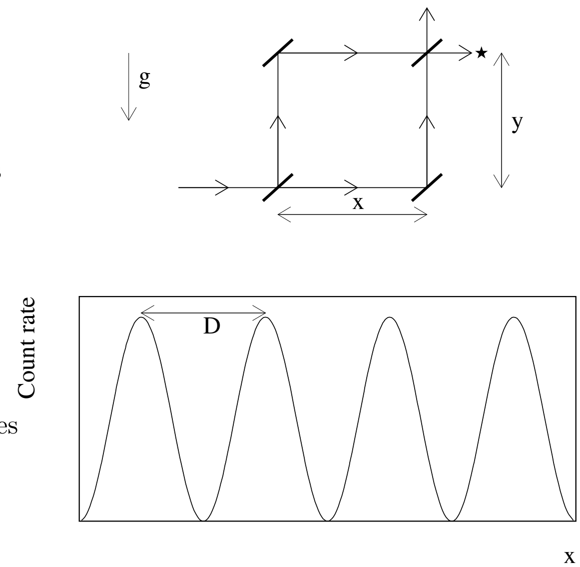

Figure 5:

4. (a) Show that for a gas sphere of constant density, the gravitational potential energy is $-(3/5)GM^2/R$ where R is the radius and M the mass.

(b) The Sun radiates energy at the rate of 3.9×10^{26} J/sec. Assuming that the Sun is a uniform sphere ($R = 7 \times 10^8$ m) with mass ($M = 2 \times 10^{30}$ kg), determine how much its radius would have to shrink each year if the radiated energy were strictly due to gravitational contraction.

(c) If this were the only source of energy for the Sun, what would the maximum present age of the Sun be, assuming its power output has been constant since its formation?

5. (a) Given that the central pressure of the Sun is 2.3×10^{16} Pa (1 Pa=1 N/m^2) and that its central temperature is 1.5×10^7 K, using the principle of hydrostatic equilibrium (pressure balances gravity) estimate the central temperature and pressure of Sirius whose mass is 2.25 M_{sun} and whose radius is 2 R_{sun}. Clearly indicate any other assumptions that you have used.

(b) From dimensional analysis and an equation of state $P \propto \rho^{5/3}$ (where P=pressure and ρ=mass density), show that white dwarfs get smaller as they get more massive.

6. A DVD has an outer diameter of 12 cm and an inner diameter of 4 cm. Holding the disk under white light you notice an iridescent rainbow of colors reflecting off the disk. Based on this give an order-of-magnitude calculation of the number of data bits encoded on the DVD.

7.

(a) A cloudless sky appears blue because particles (molecules, dust, etc...) in the atmosphere preferentially scatter shorter wavelength sunlight (Rayleigh scattering). In addition to being colored, is visible skylight polarized? If so, explain why, and describe the pattern of polarization. If on the other hand it is not polarized, explain why not.

(b) Determine the total electro-magnetic power radiated from a pulsating spherical shell of charge Q. Assume that the total charge, Q, is uniformly distributed across the surface of a sphere and that the sphere's radius is dilating according to $R(t) = A + R_0 \sin(\omega t)$, where $A > R_0$. *Hints: treat the sphere as a collection of oscillating electric dipoles. And no, this part has nothing to do with part (a).*

8. The lift force on a flying bird is $F_L \propto SV^2$, where S is the surface area of the bird's wing and V is the bird's velocity. The drag force is $F_D \propto AV^2$, where A is the frontal area of the bird into the wind.

(a) There is some minimum flying speed needed to overcome the force of gravity and keep the bird airborne. The speed scales as M^{α}, where M is the mass of the bird and α is a constant. Calculate α. (You may assume that birds of different masses have equal densities and similar body plans.)

(b) The metabolic power of a bird is observed to increase in proportion to $M^{3/4}$. Show that the power that the bird must expend to overcome the drag force increases faster than this, therefore limiting the maximum possible size of a bird capable of sustained flight.

9. A nucleon, which carries one unit of "baryon number", is a bound state of three quarks, each carrying one-third of a unit of baryon number. If one compresses a gas of nucleons to sufficiently high density, the nucleons will unbind and turn into a gas of quarks. Both quarks and nucleons are spin-1/2 particles obeying Fermi statistics.

The aim of this problem is to construct a simple model of the phase transition at zero temperature from a nucleon gas to a quark gas. In this problem neglect all internal quantum numbers of the nucleon (proton or neutron), and of the quark (color and flavor), other than spin. Also, assume that the nucleons are a non-interacting *non-relativistic* gas, and that the quarks are *fully relativistic particles* so that the kinetic energy of a quark of momentum p is cp, where c is the speed of light.

(a) Consider only zero temperature where the nucleon and quark gases are both degenerate.

1. Give the density of the nucleon gas, n_n, in terms of its Fermi momentum, p_n, and give the mean energy, E_n, per nucleon in the gas in terms of n_n.

2. Give the density, n_q, of the quark gas in terms of the quark Fermi momentum, p_q, and give the mean kinetic energy, K_q, per quark in terms of n_q.

3. If a gas of nucleons unbinds at a given fixed density, n_n, the corresponding density n_q of the quark gas equals $3n_n$. How are the nuclear and quark Fermi momenta at this density related?

(b) Assume that the quarks prefer to be bound in nucleons because the unbound state of quarks has an extra energy per unit volume, B, a positive constant. The total energy per quark in the quark gas is thus $E_q = K_q + B/n_q$.

1. Show that at low densities the nucleon gas has a lower energy per baryon than the quark gas, while at very high densities the quark gas has the lower energy per baryon.

2. Sketch the energies per baryon in the two phases as a function of baryon density, clearly labelling the curves and the baryon density, n_T, at which the system undergoes a transition from the nucleon to quark phase.

3. Write down the equation determining the transition density n_T between the two phases.