PHYSICS PH.D. COMPREHENSIVE EXAM 2006

PART II

Please do all **six** of the problems. They have equal weight. This exam is closed book. No books, notes, calculators, computers or other computational or communication devices are allowed. Please turn off your cell-phone. Writing time begins at 1pm sharp and ends at 5pm sharp, Friday, September 15, 2006.

(1) The chemical potential of a one-component fluid with N particles is given by the expression

$$\mu = \mu_0(T) + k_B T \ln\left(\frac{P}{P_0(T)}\right)$$

where T is the temperature, P is the pressure, k_B is Boltzmann's constant and the functions $\mu_0(T)$ and $P_0(T)$ are well behaved.

- Show that this system obeys the ideal gas law, $PV = Nk_BT$.
- Obtain an expression for the specific heat at constant pressure.
- Obtain the density of Helmholtz free energy $f(T, V/N)$.

Potentially useful is the Gibbs-Duhem relation $Nd\mu = VdP - SdT$.

(2) The two quantum states $|A>$ and $|B>$ form a complete set of orthonormal states of a qubit (two-level quantum system). The operator $\mathcal{R}$ represents an observable in this system.

- Find the eigenvalues of $\mathcal{R}$ if experiments determine that $< A|\mathcal{R}|A >= 1$, $< A|\mathcal{R}^2|A >= \frac{5}{4}$ and $< A|\mathcal{R}^3|A >= \frac{7}{4}$.
- Find the orthonormal eigenvectors of $\mathcal{R}$ if $< A|\mathcal{R}|B >= \frac{1}{\sqrt{8}}(1+i)$.

(3) A white dwarf star such as Sirius B is modeled as a non-relativistic quantum gas composed of completely ionized Helium. The number N of Helium atoms can be estimated by consider the approximate mass $M = 2.09 \times 10^{30} kg$ and radius $R = 5.57 \times 10^3 km$. Since there are 4 nucleons per Helium Atom, neglecting the mass of electrons we obtain $N = 1.25 \times 10^{57}$ atoms ($m_P \approx m_N \approx m = 1.67 \times 10^{-27} kg$).

- Starting from the Fermi-Dirac distribution function, obtain an expression for the Fermi energy at $T = 0$ for an ideal gas of electrons and evaluate it for the conditions on Sirius B.
- Calculate the Fermi temperature for Sirius B. Is this larger than the observed temperature of $2 \times 10^7 K$ (implying that the electron gas is degenerate)?
- Obtain an expression for the internal energy U_e of the electron gas at $T = 0K$. The total internal energy is $U = U_e + U_g$ where U_g is the gravitational internal energy of the He nuclei $U_g = -\frac{3}{5} N^2 \left(Gm^2/R\right)$.
- Find the radius which minimizes the total internal energy of the star. Calculate the radius of Sirius B and compare with the observed radius $R = 5.57 \times 10^3 km$.
- As the Helium is burned, the star begins to collapse. The electron density rises and the Fermi energy increases until it exceeds the electron mass. At this point, you must use relativistic mechanics to describe the electrons. Assuming extreme relativistic conditions, the energy and momentum are connected by $E = cp$ where c is the speed of light. Recalculate the Fermi energy for this case. Use the fact that the mean energy of the electron gas is approximately the mean momentum times c to obtain a new equation for the free energy of the star.
- Calculate the number of Helium nuclei at which the gravitational energy equals the relativistic electron energy. This is a critical number. Show that the critical mass of a star is $3.4 \times 10^{30} kg$.

The electron mass is $m_e = 9.1 \times 10^{-31} kg$, $h = 1.055 \times 10^{-34} Js$, $G = 6.67 \times 10^{-11} Nm^2/kg^2$.

(4) The Earth resembles a top whose figure axis is precessing about the normal to the ecliptic, a motion known astronomically as the precession of the equinoxes. To calculate the rate of this precession, a slight excursion into potential theory is needed to find the mutual gravitational potential of a mass point (representing the sun or the moon) and a nonspherical distrubution of matter (represneting the Earth).

(a) Find the interaction potential between the Earth of mass m and the sun (or the moon) of mass M in terms of the earth's moments of inertial I_r about the direction r separating masses m and M, and the principal moments of inertia $I_1 = I_2 \neq I_3$ of the Earth.

(b) After obtaining the potential assume that the precessional motion is much slower than the orbital motion and average over a complete orbital period to obtain an averaged potential containing only the slow (precessional) physics. What is the averaged potential?

(c) Write down the Lagrangian for the Earth's precession in terms of its principal moments of inertia.

(d) Obtain the equations of motion and calculate the precession rate. Express the precession rate in terms of the gravitational constant G, the mass of the sun (or moon) M, the principal moments of inertia and the angle of the ecliptic.

(5) Consider a particle in a state of relativistic motion under a constant force in one dimension.

  (a) Write down the Lagrangian for a relativistic particle moving under a constant force.
  (b) Obtain the equation of motion.
  (c) Solve the equation of motion. What kind of trajectory does the particle take? Compare with the nonrelativistic case.
  (d) Calculate the energy of the system.

(6) Particles of mass m are incident on a spherecally symmetric Yukawa potential

$$V(r) = V_0 \frac{e^{-\mu r}}{r}$$

where V_0 and μ are constants. To lowest order perturbation theory, show that the differential cross section for the scattering vector k is given by

$$\frac{d\sigma}{d\Omega} = \left[\frac{2mV_0}{\hbar^2(k^2 + \mu^2)} \right]^2$$

Use this result to derive the Rutherford formula for the scattering of α-particles of energy E, incident on a nucleus with atomic number Z and being scattered at an angle θ to the incident direction.