PHYS 304

MIDTERM # 2 SOLUTION

FALL 2017

1. a) Matrix elements of operator $\hat{A}$ are given by $A_{ij} = \langle i|\hat{A}|j\rangle$. Therefore we obtain

$$\hat{Q} = \mu\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \hat{P} = \varepsilon\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

b) Measurement of observable Q gives eigenvalues of $\hat{Q}$, that is $\pm\mu$. The eigenstates are $|1\rangle$, $|2\rangle$.

c) A general state $|\Psi\rangle$ can be expressed as

$$|\Psi\rangle = c_1|1\rangle + c_2|2\rangle, \qquad |c_1|^2 + |c_2|^2 = 1$$

Probabilities to measure $+\mu$ and $-\mu$ are given by

$$P_+ = |\langle 1|\Psi\rangle|^2 = |c_1|^2$$

$$P_- = |\langle 2|\Psi\rangle|^2 = |c_2|^2$$

Since the two values are measured with the same probability we must have $P_+ = P_- \Rightarrow |c_1|^2 = |c_2|^2$.
Assuming c_1 and c_2 are real, the state is

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |2\rangle\right)$$

More generally any state $|\Psi\rangle = \frac{1}{\sqrt{2}}\left(e^{i\alpha}|1\rangle + e^{i\beta}|2\rangle\right)$ with α, β real gives $P_+ = P_-$.

d) Observables whose operators commute are called compatible. They can be simultaneously measured with arbitrary accuracy. P and Q are NOT compatible because

$$[\hat{P}, \hat{Q}] = \hat{P}\hat{Q} - \hat{Q}\hat{P} = 2i\varepsilon\mu\left(|1\rangle\langle 2| + |2\rangle\langle 1|\right) \neq 0$$

$\boxed{2.}$ a) Hydrogen energy spectrum: $E_n = \frac{E_1}{n^2}$, $n = 1, 2, 3 \ldots$
$E_1 = -13.6\,eV$. The amount of energy required to ionize hydrogen in the ground state is called binding energy. Its value is $+13.6\,eV$.

b) • $$\langle r^n \rangle = \int d^3r \, r^n \, |\psi_{100}(\vec{r})|^2 =$$

$$= 4\pi \int_0^a r^2 dr \; r^n \; \frac{1}{\pi a^3} \, e^{-2r/a} \qquad y = \frac{2r}{a}$$

$$= \frac{4}{a^3} \left(\frac{a}{2}\right)^{n+3} \int_0^\infty dy \; y^{n+2} e^{-y} = \frac{1}{2}\left(\frac{a}{2}\right)^n (n+2)!$$

Take $n=1$ to obtain $\boxed{\langle r \rangle = \frac{3}{2} a.}$

• $$\langle x \rangle = \int d^3r \; x \; |\psi_{100}(\vec{r})|^2 =$$

$$= \frac{1}{\pi a^3} \int_0^\infty r^2 dr \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\varphi \, (r \sin\theta \cos\varphi) \, e^{-2r/a}$$

$$= \underline{0} \qquad \text{because} \quad \int_0^{2\pi} \cos\varphi \, d\varphi = 0$$

• The average distance from nucleus is $\underline{\langle r \rangle = \frac{3}{2} a.}$

$\boxed{3.}$ a) $A = 1$

$\Psi(t) = e^{-iE_3 t/\hbar}\,\psi_{310}$

Stationary state.

b) $1 = A^2\left(|2|^2 + |i|^2\right) \Rightarrow A = \frac{1}{\sqrt{5}}$

$\Psi(t) = \frac{1}{\sqrt{5}}\left(2e^{-iE_1 t/\hbar}\psi_{100} + ie^{-iE_2 t/\hbar}\psi_{200}\right)$

Not a stationary state

c) $1 = A^2\left(|\sqrt{5}|^2 + 1^2 + |-1|^2\right) \Rightarrow A = \frac{1}{\sqrt{7}}$

$\Psi(t) = \frac{1}{\sqrt{7}}e^{-iE_2 t/\hbar}\left(\sqrt{5}\,\psi_{210} + \psi_{211} - \psi_{21-1}\right)$

Stationary state.

d) This is not a valid initial condition. For hydrogen eigenstates ψ_{nlm} quantum number l ranges from 0 to $n-1$. Therefore ψ_{440} is not a valid eigenstate.

4. We must find the momentum-space wave function $\Phi_1(p)$ and determine its extrema.

$$\Phi_1(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx \, e^{-ipx/\hbar} \psi_1(x) \qquad \text{define } k \equiv \frac{p}{\hbar}$$

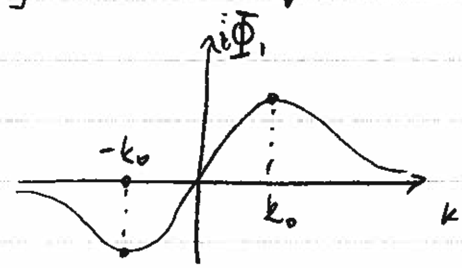
$$= \frac{A}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx \, x \, e^{-ikx} e^{-x^2/2a^2} \qquad \leftarrow \left\{ \text{use } i\frac{\partial}{\partial k} e^{-ikx} = x \right.$$

$$= \frac{A}{\sqrt{2\pi\hbar}} \left(i\frac{\partial}{\partial k}\right) \int_{-\infty}^{\infty} dx \, e^{-x^2/2a^2 - ikx} \qquad \leftarrow \left\{ \begin{array}{l}\text{complete the} \\ \text{square}\end{array} \right.$$

$$= \frac{A}{\sqrt{2\pi\hbar}} \, i\frac{\partial}{\partial k} \left[e^{-\frac{1}{2}a^2k^2} \underbrace{\int_{-\infty}^{\infty} dx \, e^{-\frac{1}{2a^2}(x+ia^2k)^2}}_{\sqrt{2\pi a^2}} \right] \qquad \leftarrow \left\{ \begin{array}{l}\text{Gaussian} \\ \text{integral}\end{array} \right.$$

Thus,

$$\boxed{\Phi_1(p) = \left(-\frac{i}{\hbar} A a^3\right) k \, e^{-\frac{1}{2}a^2k^2}}$$



Now find the extrema:

$$0 = \frac{d}{dk}\left(k e^{-\frac{1}{2}a^2k^2}\right) = e^{-\frac{1}{2}a^2k^2}\left(1 - a^2k^2\right) \quad \Rightarrow \quad k = \pm \frac{1}{a}$$

Converting back to momentum using $p = \hbar k$ the most probable values are

$$\boxed{p = \pm \frac{\hbar}{a}}$$