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Black Hole Entropy and String Theory

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Contents

In this set of lectures we will give an overview about the current knowledge concerning the origin of black hole entropy in string theory. For some general references, see [1–4].

1. Lecture 1: General Aspects of Black Holes

Motivation

Studying black holes in string theory is important as a fundamental probe of string theory. In the context of string theory the information paradox problem arising from Hawking's semiclassical calculation of the radiation spectrum of black holes is supposed to be solved because string theory is a unified quantum theory of all interactions including gravity. In this framework, a statistical mechanical explanation of black hole entropy may be possible. The AdS/CFT correspondence is a conjecture in which we have an explicit realization in principle of the information retention scenario.

The study of black holes is also important in applications of string theory, in particular in the application of the AdS/CFT correspondence to strongly coupled thermal gauge theories. These investigations already have led to many new recent developments, and we can expect more.

General Aspects of Black Holes

We start with the Schwarzschild black hole metric in four dimensions,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 d\Omega_2^2 \,. \tag{1.1}$$

In this metric the timelike Killing vector is $\frac{\partial}{\partial t}$. This metric is the metric of a static black hole. If we consider stationary black holes whose metric is time independent, but generically has cross terms $d\phi dt$ corresponding to rotation stationary black, the timelike Killing vector is generalised to

$$\varepsilon = \frac{\partial}{\partial t} + \Omega_H \frac{\partial}{\partial \phi}\,, \tag{1.2}$$

where Ω_H is the angular velocity of the horizon. Rotating stationary black holes have axial symmetry, i.e. another spacelike Killing vector associated with an angular coordinate, and furthermore, in four dimensions all known stationary black holes are at least axisymmetric.

A horizon is defined by null hypersurface, i.e. a hypersurface defined for example by an implicit equation $f(x^\mu) = 0$, with a normal vector $n^\mu = \nabla^\mu f$ which is light-like, $n_\mu n^\mu = 0$. In this case n^μ is also tangent to the surface. If choosing lightcone coordinates u, v around the horizon (put at $u = 0$, see figure 1), the unit normal vector would be $n^u = 1\,, n^i = 0\,, i \neq u$. The vector tangential to the horizon would be $t^v = 1\,, t^i = 0\,, i \neq v$. Locally around the horizon the metric would contain a piece $\mathrm{d}u\mathrm{d}v$, which implies $n_\mu t^\mu = 0$.

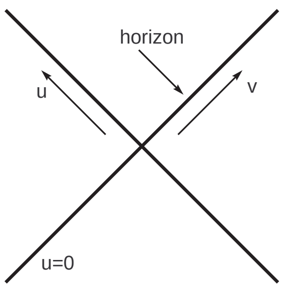

Figure 1: Null Surface and Light Cone Coordinates

For all known stationary black holes the horizon is a Killing horizon, i.e. the normal vector coincides with the timelike Killing vector, evaluated at horizon. Let ε^μ be the timelike Killing vector, then the horizon is at

$$\varepsilon^\mu \varepsilon_\mu = 0\,. \tag{1.3}$$

ε can be used to define a quantity measuring the strength of gravity at the horizon, called surface gravity. The surface gravity κ is defined as the covariant derivative of the norm of ε, evaluated at the horizon,

$$\nabla_\mu(\varepsilon^\nu \varepsilon_\nu) = -2\kappa\varepsilon_\mu\,. \tag{1.4}$$

As $\varepsilon_\mu \varepsilon^\mu$ is constant along the horizon, its gradient (the covariant derivative), must point along the unit normal vector to it, which is ε^μ itself. The factor of -2 is for mere convenience. The surface gravity is in turn related to the Hawking temperature of the horizon by

$$T_H = \frac{\kappa}{2\pi}\,. \tag{1.5}$$

For the Schwarzschild black hole (1.1) one finds $\kappa = (4MG)^{-1}$ and $T_H = (8\pi MG)^{-1}$. We need to choose smooth coordinates at horizon to evaluate κ. Once T_H and Ω_H are known, the entropy is required to satisfy the first law of (black hole) thermodynamics,

$$\delta M = T_H \delta S + \Omega_H \delta J \;\Leftrightarrow\; \delta S = \frac{1}{T_H}\delta M - \frac{\Omega_H}{T_H}\delta J\,. \tag{1.6}$$

For the nonrotating Schwarzschild black hole (1.1), $\Omega_H = 0$, so the entropy $S = \frac{4\pi M^2}{G} = \frac{A_H}{4G}$ actually fulfills (1.6).

Also, note that in stringy microscopic entropy computations it is not at all manifest why entropy S should have anything to do with the horizon area. One way to show that the entropy localizes to a term on the horizon is to use the Euclidean action. Let's just consider the Schwarzschild solution (1.1) for simplicity. We thus take

$$t \to i\tau\,, \quad \tau \cong \tau + \beta\,, \quad \beta = \frac{1}{T_H} \tag{1.7}$$

to avoid conical singularity at the horizon. These identifications come about as follows: After the Wick rotation, the $r-\tau$-part of the euclidean metric is

$$f(r)\mathrm{d}\tau^2 + \frac{\mathrm{d}r^2}{f(r)}.$$

Expanding the Killing norm near the horizon

$$f(r) = f'(r_H)(r - r_H)\,,$$

and introducing a new coordinate

$$R^2 = \frac{4}{f'(r_H)}(r - r_H)\,,$$

the $r-\tau$-part of the metric becomes

$$\left(\frac{f'(r_H)}{2}\right)^2 R^2\mathrm{d}\tau^2 + \mathrm{d}R^2\,.$$

This is the standard metric on $\mathbb{R}^2$, provided we identify the (angular) variable τ periodically,

$$\tau \sim \tau + \beta\,, \quad \beta = \frac{1}{T_H} = \frac{4\pi}{f'(r_H)}\,.$$

In the semiclassical approximation, the euclidean path integral (the black hole partition function) is dominated by

$$Z = \mathrm{Tr}e^{-\beta H} = e^{-I_E = }e^{S-\beta M}\,, \tag{1.8}$$

i.e. $S = -I_E + \beta M$, where I_E is the Euclidean action. To see why S is related to the horizon, we consider the $\rho-\tau$ plane, which has topology $\mathbb{R}^2$ (see fig. 2), with $\rho = r - 2M$, going to zero at the horizon.

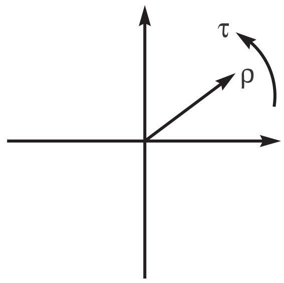

Figure 2: Near-Horizon Topology of Euclidean Schwarzschild BH

We need to compute the euclidean on-shell action. For this purpose, we introduce an artificial boundary of space-time, as otherwise on-shell actions might diverge when integrated over infinite space. The Einstein-Hilbert action with boundary terms has the form

$$I = \frac{1}{16\pi G}\int\limits_{\mathcal{M}} d^D x\sqrt{g}\, R - \frac{1}{8\pi G}\int\limits_{\partial\mathcal{M}} d^{D-1}x\sqrt{h}\, Tr\mathcal{K} + S_{\partial\mathcal{M}}(h)\,. \tag{1.9}$$

The second term is the York-Gibbons-Hawking term needed for a well-defined variational principle such that the metric on the boundary $\partial\mathcal{M}$ fulfills Dirichlet boundary conditions. The extrinsic curvature of the boundary is the covariant derivative of an outward pointing unit normal on $\partial\mathcal{M}$,

$$\mathcal{K}_{\mu\nu} = \nabla_\mu \backslash_\nu + \nabla_\nu \backslash_\mu\,, \quad \mathrm{Tr}\mathcal{K} = \mathcal{K}_\mu{}^\mu\,. \tag{1.10}$$

The last term includes counterterms to cancel possible (large volume) divergences. It only depends on the induced metric h at the boundary[1], and thus does not contribute to the equations of motion. Suppose we only integrate over a wedge of $\Delta\tau$, see fig. 3. The solution is static, i.e. time independent. Imposing Dirichlet boundary conditions for the boundary metric is equivalent to choosing the metric as canonical coordinate q, and thus staticity implies that $\dot{q} = 0$, as well as the on-shell Hamiltonian H being time-independent (equal to the mass), and thus

$$I_{wedge} = \int d\tau(p\dot{q} + H) = M\Delta\tau\,. \tag{1.11}$$

Also note that we have to remember to include boundary terms, including one at tip of wedge. If taking $\Delta\tau = \beta$, i.e. integrating once around the τ-circle, there is the no boundary at tip since the full space is smooth. So we should subtract the Gibbons-Hawking term at the tip

$$I_{BH} = M\beta - I_{GH}|_{hor}\,. \tag{1.12}$$

Comparing with (1.8) we find a connection between entropy and on-shell boundary action,

$$S = I_{GH}|_{hor}\,. \tag{1.13}$$

For the Schwarzschild black hole we can easily calculate the extrinsic curvature $\mathcal{K}$. The Gibbons-Hawking term is then given by

$$I_{GH} = \frac{1}{8\pi M}(4\pi)(\beta)\sqrt{1-\frac{2M}{r}}\,\partial_r\left(r^2\sqrt{1-\frac{2M}{r}}\right)\Big|_{r=2M} = \frac{\beta M}{2G} = \frac{A}{4G}\,. \tag{1.14}$$

[1] In general it will depend on the boundary value of fields which satisfy Dirichlet boundary conditions

All that was said above holds for two-derivative gravity. In general, there will be a more complicated version of I_{GH}, which however still clearly leads to S as an integral over horizon. The general result for a difeomorphism invariant theory is given by Wald's formula [5] (see also the review [6]),

$$S = 2\pi \int_{\text{Horizon}} \frac{\delta \mathcal{L}}{\delta R_{\mu\nu\rho\sigma}} \epsilon^{\mu\nu} \epsilon^{\rho\sigma} \sqrt{h}\, d\Omega \,, \tag{1.15}$$

where $\epsilon^{\mu\nu}$ is the binormal to the horizon, i.e. $\epsilon = du \wedge dv$ in the near horizon limit. This formula obeys the first law of thermodynamics (1.6) by construction. One can also show that it agrees with the Euclidean path integral approach. Fir extremal black holes, i.e. with a near horizon geometry containing an AdS_2 factor, it can be shown to be equivalent to Sen's entropy function [7]. Note, however, that this formula is not valid for theories with gravitational Chern-Simons terms, where the Lagrangian density itself is not diffeomeorphism invariant, but only the action is.

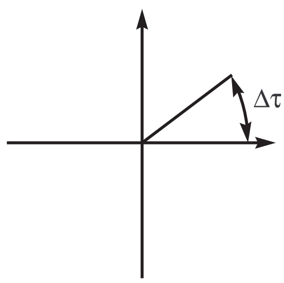

Figure 3: Integrating over a Wedge

Attractor Mechanism

A central tool for studying entropy of extremal black holes is the attractor mechanism. In general, an extremal black hole is specified by its charges, as well as the values of massless scalar fields at infinity. Generically, these scalar fields have nontrivial dependence as one flows towards the horizon. For extremal black holes two things can happen: Some of the scalar fields in the theory might flow to values specified by the charges only. Thus the black hole entropy will not depend on asymptotic values of these scalars, which is a very nontrivial statement and is tightly connected to the extremality of the solutions under consideration. For other scalars which are not fields like this (e.g. a massless non-coupled scalar) one can show that the entropy is independent of their values at the horizon, and hence at infinity. So in general,

$$S_{BH} = S_{BH}(Q_I, \phi^I) \,, \tag{1.16}$$

where Q_I are charges, and ϕ^I are scalar fields. Intuitively, if the ϕ dependence at infinity would enter one could use this to violate the second law of thermodynamics, by imagining physical processes like a pulse of the scalar fields under consideration falling into the black hole which changes the scalar value at infinity while decreasing the entropy. The statement of the **Attractor Mechanism** is that the entropy of a black hole does only depend on the charges of the black hole, and not on the value of the scalar fields at all points in space-time.

Intuitively, this can be seen as a consequence of "no-hair" theorems: A classical black hole without hair is only specified by it's charges, and thus the entropy (being a semiclassical quantity), should also be so.

In more detail these results are establishes as follows: Extremal black holes have near horizon geometries, obtained by scaling $r \to \epsilon r$. In a wide variety of cases one can show this geometry has an AdS_2 factor (i.e. has $SO(2,1)$ symmetry). In four dimensions, for example, the near-horizon geometry is always $AdS_2 \times S^2$. In favorable cases there a local AdS_3 factor. On the AdS_2 scalars are constant by symmetry, 2-form field strengths take the form

$$F_I = \phi_I \epsilon_{AdS} \quad (\textit{electric potential})\,, \tag{1.17}$$

and given some compact space $\mathcal{M}_p$, which is generically present in the near-horizon geometries, a p-form field strength can gave

$$F^I = P^I \epsilon_{\mathcal{M}_p} \quad (\textit{magnetic charges})\,. \tag{1.18}$$

Here ϵ_{AdS} and $\epsilon_{\mathcal{M}_p}$ denote the volume forms on the AdS factor and on the compact space, respectively. One then evaluates the Lagrangian density with fixed ϕ_I, P^I on the near-horizon solution, with other data unspecified except through symmetry constraints. This defines a function f via

$$I|_{\mathbf{onshell}} = \int \mathrm{d}^2 x f\,.$$

Sen's entropy function is then the Legendre transform of f w.r.t. the electric charges ϕ_I,

$$S = \phi_I \frac{\partial f}{\partial \phi_I} - f\,.$$

From this function, equations of motion for the electric and magnetic charges, the scalars and the relative radii of the AdS part and the compact part of the near-horizon geometry can be derived by the usual Euler-Lagrange method. Upon solving them, it can be shown that the extremum of S just yields Wald's entropy (where this is applicable),

$$S_{\text{extremum}} = S_{\text{Wald}}\,.$$

The equations of motion either fix the other data in terms of ϕ_I, P^I, or leave them unspecified. Unfixed ones do not appear in the action density and so will not affect entropy. Electric charges are related to ϕ_I by

$$Q^I \sim \frac{\partial f}{\partial \phi^I}\,,$$

so solutions are fixed by Q^I, P^I. Then we have $S_{BH}(Q^I, P^I)$ by Wald's formula, which proves the Attractor mechanism.

Due to this result, one can determine S_{BH} in region where the two-derivative approximation is valid, in terms of the electric and magnetic charges. One can then adjust the asymptotic moduli to a weak string coupling region, corresponding to microscopic black holes (in general strong couplings imply large moduli and thus macroscopic black holes). But as we know now, the entropy does not depend on the moduli (scalar fields), but only on the charges, and thus the macroscopic black hole entropy can be interpolated to microscopic regimes, at least for extremal black holes. In the microscopic regime the full geometry gets highly curved, but unless there is some sort of a phase transition, the entropy formula will continue to hold. In week coupling region it can be directly compared with a microscopic counting, if available. Note that this line of argument does not use supersymmetry, so holds for non-BPS extremal black holes. In this case, other quantities like the mass do change as one adjusts the moduli, but entropy doesn't. For the extremal black holes encountered in string theory this formula simplified a lot, as we will see. For the nonextremal case, however, nothing prevents the entropy to depend on the asymptotic moduli. However, for the special case of the enlarged near-horizon isometry AdS_3, it was possible to extend Sen's argumentation to near-extremal black holes.

2. Lecture 2: Black Hole Solutions to Five-Dimensional Supergravity

Overview of Relevant Black Hole Solutions

Black hole entropy is understood in string theory for asymptotically flat black holes preserving some supersymmetry (BPS black holes), as well as for some non-BPS and nonextremal black holes related to the BPS ones. BPS black hole solutions are known in four and five dimensions. In general, not much is known about the solution space of black holes in more than five dimensions. For $D \leq 5$ there are various tricks for finding BPS solutions available which do not extend to $D \geq 6$. In $D = 5$, one can have BPS black holes, black strings and black rings. In $D = 4$ the situation is less rich: there are only non-rotating BPS black holes, as well as multi-centered versions of those. Thus five dimensions are the most fruitful arena to work, since it turns out that all four-dimensional BPS solutions can then just be found by dimensional reduction (possibly on Taub-NUT). In this lecture we will thus discuss the relevant $D = 5$ supergravity and its solutions.

A nice way to think about $D = 5$ supergravity is as M-theory compactified on a six-dimensional manifold $\mathcal{M}_6$. Different manifolds will preserve different amounts of supersymmetry, see table 1. In eleven dimensions, there are M5-branes and M2-branes, which can be wrapped on $\mathcal{M}_6$ to get charged objects in $D = 5$. The supersymmetry-preserving possibilities are

- M2-branes wrapped on two-cycles, yielding a particle in five dimensions and
- M5-branes wrapped on four-cycles, yielding strings in five dimensions.

$\mathcal{M}_6$	T^6	$\mathrm{K3} \times \mathrm{T}^2$	CY_3
$\mathcal{N} =$	8	4	2

Table 1: Compactifications of M-theory and Preserved Supersymmetries

We will thus be able to construct charged black hole and black string solutions. The black rings can then be obtained by bending the string int o a circle and stabilizing it by giving it some angular momentum, see fig. 4.

Five-dimensional Supergravity from Eleven Dimensions

To obtain the five-dimensional supergravity action, we first have to do a Kaluza-Klein reduction of eleven-dimensional supergravity, which is straightforward. We start with the eleven-dimensional supergravity action,

$$S_{11} = -\frac{1}{2\kappa_{11}^2} \int \mathrm{d}^{11}x\sqrt{-g} \left(R^{(11)} + \frac{1}{2}|F_4|^2 \right) + \frac{1}{12\kappa_{11}^2} \int A_3 \wedge F_4 \wedge F_4 \,, \tag{2.1}$$

which is completely fixed by $\mathcal{N} = 1$ supersymmetry in $D = 11$. $F_4 = \mathrm{d}A_3$ is the field strength of the three-form gauge field, and $|F_4|^2 = F_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma}$. $R^{(11)}$ is the eleven-dimensional Ricci scalar. Recall that electric and magnetic charges are defined by integrating over spheres of appropriate dimensionality,

$$Q_{\mathrm{el}} = Q_{\mathrm{M2}} \propto \int_{S^7} *_{11} F_4 \,, \tag{2.2}$$

$$Q_{\mathrm{mag}} = Q_{\mathrm{M5}} \propto \int_{S^4} F_4 \,, \tag{2.3}$$

with $*_{11}$ being the eleven-dimensional Hodge star. To reduce on the compact manifold $\mathcal{M}_6$, we split the metric naturally into

$$\mathrm{d}s_{11}^2 = \mathrm{d}s_5^2 + \mathrm{d}_{\mathcal{M}_6}s^2 \,. \tag{2.4}$$

Besides the five-dimensional metric, we will find a number of gauge fields and scalar fields from the reduction. For the generic $\mathcal{N} = 2$ case, the gauge fields come from reduction of A_3, and the scalars come from the CY_3 moduli, of which there are two types,

- Kähler moduli, which combine into $D = 5$ $\mathcal{N} = 2$ vector multiplets and
- complex structure moduli, which yield $D = 5$ $\mathcal{N} = 2$ hypermultiplets.

The hypermultiplet scalars are normally neglected in black hole physics. At least in two-derivative gravity they are just constants by no-hair theorems, and thus decouple effectively, a fact which is also eludicated by the attractor mechanism. We will thus neglect them in the following. One example is the overall size of $\mathcal{M}_6$, which we will just set to be a fixed constant, which we choose to be $\mathrm{Vol}(\mathcal{M}_6) = 1$ in units of $\kappa_{11}^2 = 2\pi^2$.

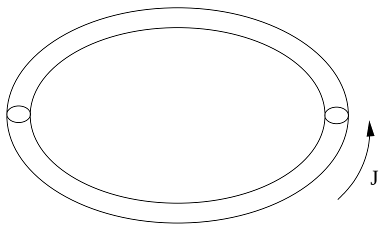

Figure 4: Black Ring Stabilised by Rotation

To carry on, we expand the Kähler form J on $\mathcal{M}_6$ in a complete basis of $(1,1)$-forms $\{J_I\}$,

$$J = \sum_I M^I J_I\,, \quad I = 1, \ldots, h^{(1,1)}\,. \tag{2.5}$$

M^I are then the real Kähler moduli. The three-form gauge field can be expanded as

$$A_3 = \sum_I A^I \wedge J_I\,, \tag{2.6}$$

with A^I being five-dimensional gauge fields. The four-form field strength decomposes as

$$F_4 = \sum_I F^i \wedge J_I\,, \tag{2.7}$$

yielding a collection of $U(1)$ gauge fields in five dimensions. The eleven-dimensional Chern-Simons term then reduces as

$$\int A_3 \wedge F_4 \wedge F_4 = \int_{\mathcal{M}_6} J_I \wedge J_J \wedge J_K \int_{\mathcal{M}_5} A^I \wedge F^J \wedge F^K = C_{IJK} \int_{\mathcal{M}_5} A^I \wedge F^J \wedge F^K\,,$$

with C_{IJK} being the "triple intersection numbers". Three two-cycles in a six-dimensional manifold generically intersect in a finite number of distinct points, which are counted by C_{IJK}. Also, since we set the volume of the internal manifold to one and the volume form is given by $J \wedge J \wedge J$, we enforced the "real special geometry constraint" on the Kähler moduli,

$$\mathrm{Vol}(\mathcal{M}_6) = \frac{1}{3!} \int_{\mathcal{M}_6} J \wedge J \wedge J = \frac{1}{3!} C_{IJK} M^I M^J M^K = 1\,. \tag{2.8}$$

It is straigtforward to show that the kinetic term for the five-dimensional gauge fields is generated by $|F_4|^2$, and the kinetic terms for the Kähler moduli comes from $R^{(11)}$. Putting all the pieces together, one arrives at the $D = 5$ $\mathcal{N} = 2$ action ($\kappa_{11} = 2\pi^2$)

$$\begin{aligned}
S_5 &= -\frac{1}{4\pi^2}\int \mathrm{d}^5x\sqrt{|g|}\left(R + G_{IJ}\partial_a M^I \partial^a M^J + \frac{1}{2}G_{IJ}F^I_{ab}F^{Jab}\right) \\
&\quad + \frac{C_{IJK}}{24\pi^2}\int\limits_{\mathcal{M}_6} A^I \wedge F^J \wedge F^K\,, \\
G_{IJ} &= \frac{1}{2}\int\limits_{\mathcal{M}_6} J_I \wedge *_6 J_J = \frac{1}{2}\left(M_I M_J - M_{IJ}\right)\,, \\
M_I &= \frac{1}{2}C_{IJK}M^J M^K\,, \quad M_{IJ} = C_{IJK}M^K\,.
\end{aligned} \tag{2.9}$$

Eq. (2.9) is the action of $D = 5$ $\mathcal{N} = 2$ supergravity coupled to an arbitrary number of abelian vector multiplets. Note that the fact that the metric on the scalar manifold, G_{IJ}, is the same metric contracting the kinetic terms for the gauge fields, is forced upon us by supersymmetry. It is also interesting to note that the whole action (2.9) can be derived as the supersymmetric completion of the Chern-Simons term. As stated earlier, all known asymptotically flat BPS black hole solutions can be embedded in this theory. Standard examples like the D1-D5-P black hole can be obtained from these black holes by U-duality. Note also that for the $\mathcal{N} = 4, 8$ cases, one sets extra gravitino multiplets flat that need to be taken into account.

BPS Black Hole Solutions

We now start searching for BPS black hole solutions. In $D = 5$ there are electric charges

$$q_I = \frac{1}{2\pi^2}\int\limits_{S^3} G_{IJ} *_5 F^J\,, \tag{2.10}$$

and magnetic charges

$$p^I = \frac{1}{2\pi^2}\int\limits_{S^2} F^I\,. \tag{2.11}$$

Note that without the G_{IJ}, q_I would not be a conserved charge. As we will see later, the electric charges will be carried by electrically charged black holes, and the magnetic charges by black strings. BPS solutions carrying these charges can be constructed in an elegant fashion, following Gauntlett et. al. [8], which will be sketched now.

A general $\mathcal{N} = 2$ BPS solution preserves half of the supersymmetries, i.e. $\mathcal{N} = 1$, and thus must possess a globally defined, nowhere vanishing Killing spinor ε, from which we can form a vector $\bar{\varepsilon}\gamma^a\varepsilon$. This vector can be shown to be a Killing vector, either timelike or null. The analysis of solutions is different depending on the timelike or null condition. We will for now restrict ourselfes to the **timelike case**, which yields black hole and black ring solutions. The null case leads to magnetically charged black string solutions.

A general metric with a timelike Killing vector can always be written in the form

$$\mathrm{d}s_5^2 = e^{4U_1(x)} (\mathrm{d}t + \omega)^2 - e^{-2U_2(x)} \mathrm{d}s_B^2(x)\,, \tag{2.12}$$

where we have chosen the time coordinate such that the timelike Killing vector is $\frac{\partial}{\partial t}$. We thus splitted the five-dimensional space into time and a four-dimensional base space B with coordinates x and metric $\mathrm{d}s_B^2$. $U_{1,2}$ are functions on the base, and ω is a one-form on B. Supersymmetry restricts the base B to be a Hyper-Kähler manifold, and enforces the condition

$$U_1 = U_2 = U\,.$$

Because of the Killing symmetry, the moduli can only depend on the base coordinates, $M^I(x)$, and the field strengths can be written as

$$F^I = \mathrm{d}\left[M^I e^{2U} (\mathrm{d}t + \omega)\right] + \Theta^I\,. \tag{2.13}$$

For the field strength to fulfill the Bianchi identity $\mathrm{d}F^I = 0$, the Θ^I have to be closed two-forms on the base B. With these definitions, the BPS equations reduce to the following three conditions:

$$\Theta^I = - *_4 \Theta^I\,, \tag{2.14}$$

$$\nabla^2 \left(M_I e^{-2U}\right) = \frac{1}{2} C_{IJK} *_4 \left(\Theta^J \wedge \Theta^K\right)\,, \tag{2.15}$$

$$\mathrm{d}\omega - *_4 \mathrm{d}\omega = -e^{-2U} M_I \Theta^I\,. \tag{2.16}$$

These equations have to be supplemented by the real special geometry condition (2.8). After a choice of a Hyper-Kähler base manifold B, Eqs. (2.14)-(2.16) define a system of linear equations with sources: Finding an anti-selfdual closed two-form on B is a linear problem, as well as solving the Laplace equation (2.15) on B with fixed right-hand side (∇ is the covariant derivative on the base). One can then use the solutions of (2.14) and (2.15) to fix the right-hand side of (2.16) and solve for ω, which is again a linear equation (in ω). This is of course to be expected, as BPS equations generally are first-order linear partial differential equations which imply the equations of motion of a system.

Given the structure of the BPS equations, it is not surprising that they can be solved in terms of harmonic functions. The simplest example directly yields an **electrical charged nonrotating black hole**: We require an electric field strength component to be switched on, but Θ^I are two-forms on the base and thus can only give magnetic components to F^I, so we can simply set $\Theta^I = 0$. Furthermore, if ω would be present, it would induce angular momentum in the black hole metric through $\mathrm{d}t \otimes \omega$ terms, so we also set $\omega = 0$. Setting both forms to zero is possible for example for the choice $B = \mathbb{R}^4$. In that case, we simply set

$$\mathrm{d}s^2_{\mathbb{R}^4} = \mathrm{d}r^2 + r^2 \mathrm{d}\Omega_3^2 \,.$$

What remains to be solved is the Laplace equation (2.15), whose right-hand side vanishes now. It thus can be solved in terms of harmonic functions on $\mathbb{R}^4$,

$$M_I e^{-2U} = H_I = h_I + \frac{q_I}{r^2} \,. \tag{2.17}$$

To write down the black hole metric, one must invoke the real special geometry constraint (2.8), which makes it possible to solve for U and M^I separately. However, one first needs to find the M^I in terms of the M_I by inverting the definition $M_I = \frac{1}{2} C_{IJK} M^J M^K$. Generically, this can not be done explicitly. An exception is T^6, where the only non-vanishing triple intersection numbers are $C_{123} = 1$ and permutations thereof. In this case, the solution is simply

$$M^I = \frac{1}{M_I} \,,$$

and the special geometry constraint becomes

$$M^1 M^2 M^3 = 1 \,.$$

In this case one finds the moduli in terms of the harmonic functions to be

$$e^{-6U} = H_1 H_2 H_3 \,, \quad M^I = \frac{e^{-2U}}{H_I} = \frac{(H_1 H_2 H_3)^{\frac{1}{3}}}{H_I} \,.$$

The metric turns out to be like that of a five-dimensional Reissner-Nordstrom black holoe,

$$ds^2 = e^{4u(r)} dt^2 - e^{-2u(r)} \left(dr^2 + r^2 d\Omega_3^2 \right) . \tag{2.18}$$

The horizon is located at $r = 0$, and $e^{2u(r)} \sim \frac{r^2}{l^2}$. The value of l is given as follows. Given q_I, we define q^I via

$$\frac{1}{2} C_{IJK} q^J q^K = q_I . \tag{2.19}$$

Then we have

$$l^3 = \frac{1}{3!} C_{IJK} q^J q^J q^K . \tag{2.20}$$

The black hole entropy is given by

$$S = \frac{l^3 \Omega_3}{4 G_5} = 2\pi l^3 \,. \tag{2.21}$$

The moduli asymptotic value of M_I goes between $M_I = h_I$ at infinity and $M_I = \frac{q_I}{l^2}$ at the horizon. This is an example of the attractor mechanism: The near horizon values are independent of asymptotic values. Also, the entropy only depends on q_I, not the asymptotic moduli.

Note that to preserve asymptotic flatness, the boundary conditions for the Laplace problem have to be chosen such that $M_I e^{-2U}$ approaches a constant as $r \to \infty$. Also, even in this simply example, one finds the attractor mechanism at work: At infinity, the moduli M^I approach some constant values, which can be chosen freely, but they all flow to a fixed value $M^I = (q_1 q_2 q_3)^{\frac{1}{3}}/q_I$ at the origin $r = 0$, where the horizon is situated. The near-horizon values of the scalar fields are thus fixed by the charges of the black hole, not by its values at infinity. The near horizon ($r \to 0$) geometry of this black hole is

$$\mathrm{d}s^2 = \frac{r^4}{(q_1 q_2 q_3)^{\frac{2}{3}}}\mathrm{d}t^2 - \frac{(q_1 q_2 q_3)^{\frac{1}{3}}}{r^2}\mathrm{d}r^2 + (q_1 q_2 q_3)^{\frac{1}{3}}\mathrm{d}\Omega_3^2\,,$$

which becomes $AdS_2 \times S^3$ after a redefinition $r^2 = \rho$. The AdS radius is related to the three-sphere radius as $l_{AdS_2} = l_{S^3}/2$, and the whole geometry is independent of the moduli at infinity, again confirming the attractor mechanism.

By turning on the one-form ω while keeping $\Theta^I = 0$, one can readily add angular momentum and thus find **rotating charged black holes**. In general, by (2.16), one has to solve $\mathrm{d}\omega = *_4\mathrm{d}\omega$ on the chosen Hyper-Kähler base B. In the above case $B = \mathbb{R}^4$, the straightforward solution is

$$\mathrm{d}\omega = J(\mathrm{d}x^1 \wedge \mathrm{d}x^2 + \mathrm{d}x^3 \wedge \mathrm{d}x^4)\,,$$

with J being the amount of angular momentum. In this case, the rest of the above analysis, in particular the spherical symmetry and the values for the moduli and $U(x)$, stay unchanged.

From the area of their event horizons, the entropy of both the nonrotating and the rotating black hole are found to be ($G_5 = \pi/4$)

$$S = 2\pi\sqrt{q_1 q_2 q_3 - J^2}\,.$$

Albeit of both solutions being supersymmetric and thus extremal, there is a maximal value $J_{max} = \sqrt{q_1 q_2 q_3}$ for the angular momentum, above which the solution develops closed timelike curves. For a general CY_3 fold without rotation, one finds a generalised formula

$$S_{\mathrm{CY}_3} = 2\pi\sqrt{\frac{1}{3!}C_{IJK}q^I q^J q^K}\,.$$

Albeit of the rather straightforward construction of these solutions, the microscopic understanding for the general CY_3-fold case in terms of string theory is still not known, except of tractable examples with a high amount of supersymmetry such as T^6 (see later).

Another, more general class of base manifolds which can be treated this way are **Gibbons-Hawking type** bases, defined by

$$\mathrm{d}s_B^2 = \frac{1}{H^0(\vec{x})}\left(\mathrm{d}x^5 + \chi\right)^2 + H^0(\vec{x})\mathrm{d}\vec{x}^2\,, \tag{2.22}$$

$$\vec{\nabla}\times\vec{\chi} = \vec{\nabla}H^0 \Rightarrow 0 = \vec{\nabla}(\vec{\nabla}\times\vec{\chi}) = \vec{\nabla}^2 H^0(\vec{x})\,. \tag{2.23}$$

Here $x^5 \sim x^5 + 4\pi$ is the fourth coordinate of the base B, $\vec{x} \in \mathbb{R}^3$, and χ is a one-form on the base. By the condition (2.23), which follows from Ricci-flatness of B, the function H^0 has to be harmonic (with isolated singularities). In fact, requiring $U(1)\times U(1)$ symmetry, the base must be Gibbons-Hawking. If one thinks about x^5 as the Kaluza-Klein circle, χ can be viewed as the gauge field after KK reduction, and eq. (2.23) then shows that the singularities of H^0 correspond to magnetic monopoles, so-called "**Kaluza-Klein monopoles**". There are several special cases:

1. $H^0 = 1$, $\chi = 0$ yielding $\mathbb{R}^3 \times S^1$,
2. $H^0 = \frac{1}{|\vec{x}|}$, $\chi = 0$, blowing up at $\vec{x} = 0$, which can be compensated by sending $R_{x^5} \to 0$, yielding $\mathbb{R}^4$,
3. $H^0 = \frac{p}{|\vec{x}|}$ gives $\mathbb{R}^4/\mathbb{Z}_p$.
4. $H^0 = 1 + \frac{p}{|\vec{x}|}$ yields "Taub-NUT" spaces, which interpolate between $\mathbb{R}^3 \times S^1$ at $|\vec{x}| \to \infty$ and $\mathbb{R}^4/\mathbb{Z}_p$ at small $|\vec{x}|$. They have a cigar-shaped geometry (see fig. 5), with a $\mathbb{Z}_p$ singularity at the origin.

More general H^0 correspond to multi-center black holes, with the general rule that a singularity of the harmonic function H^0 corresponds to a center of a particular black hole. In general, if H^0 is non-zero at infinity, the asymptotic x^5 circle is of finite size and thus the geometry is (including time) asymptotic to $\mathbb{R}^{3,1} \times S^1$, yielding four-dimensional asymptotically flat black holes after reduction on the S^1. For five-dimensional asymptotics, H^0 should tend to zero as $|\vec{x}| \to \infty$.

Note that the microscopic understandings of these black holes is quite poor, i.e. there is no well-defined computation giving the entropy in general. There is an AdS_2 factor in the near-horizon geometry, leading to a possible conformal quantum mechanics on the

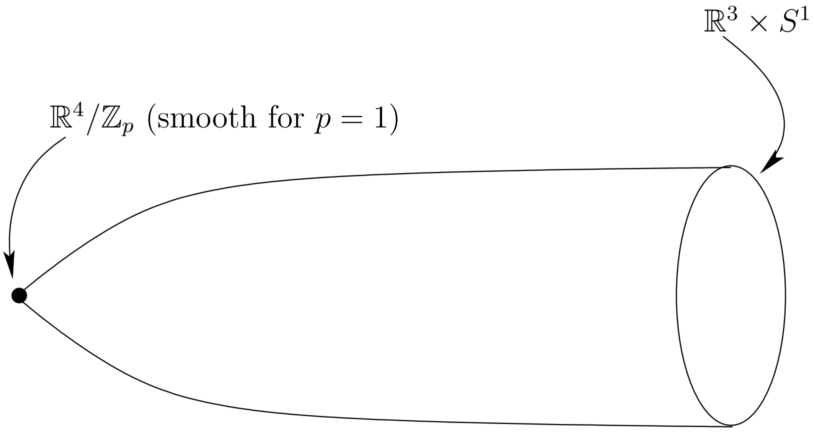

Figure 5: Taub-NUT as an interpolating cigar.

boundary, but this has never been made precise. Or perhaps one should interpret the $SL(2,R)$ isometry group as matching that of a purely chiral CFT (see work of Sen and Strominger during the past few months).

Suppose we consider the T^6 or $K3 \times T^2$ cases with $N = 8, 4$ SUSY. In this case we are in both shape. We can reduce $\mathcal{M}$-theory on a circle $S^1 \subset T^2$ to get type IIA string theory, and then further dualize to IIB, living on $T^4 \times S^1$ or $K3 \times S^1$ with the charges of a $D1 - D5 - P$ system. This is the classic Strominger-Vafa setup [13]. The solution in this frame has a near horizon geometry $AdS_3 \times S^3 \times (T^4 \text{or } K^3)$. The AdS_2 combines with some of the T^2 circles to give AdS_3 These black holes can be well understood microscopically via $D1 - D5$ CFT. The AdS_3 factor is the main helping factor, but for $5D$ $\mathcal{N} = 2$ black hole no such help is available.

Other solutions

5D spinning BMPV black hole

We can easily make the black hole rotating by changing $H_0 = 0 \rightarrow \frac{J}{8|\vec{x}|}$. This gives the BMPV black hole [14]. Note that the spatial rotation group in $5D$ is $SO(4) = SU(2)_L \times SU(2)_R$, and the BMPV solution has $J_L^3 = J, J_R^3 = 0$. The entropy is $S = 2\pi\sqrt{l^6 - J^2}$ with l the same as in the static case. Again we have a near horizon $AdS_2 \times S^2$ geometry.

BMPV black hole on Taub-NUT

We can add a $Taub - NUT$ term to BMPV black hole by taking

$$H^0 = h^0 + \frac{p^0}{|\vec{x}|}, \quad H_0 = h_0 + \frac{J}{8|\vec{x}|} \tag{2.24}$$

This gives a rotating black hole on top of a Taub-NUT space. Reducing along the fifth-dimensional S^1 S^1 gives a 4D black hole with electric charges q_I. Also, the angular mo-

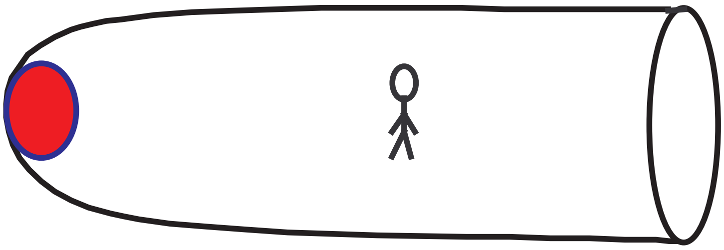

Figure 6: Black Hole in a small Taub-NUT space.

mentum is given by $q_0 = 2J$ units of KK electric charge. p^0 becomes KK magnetic charge. The entropy is given by

$$S = 2\pi\sqrt{p^0 l^6 - \frac{1}{4}(p^0 q_0)^2}\,. \tag{2.25}$$

Note that this solution is a nice illustration of the **4D-5D connection**. By varying h^0 we can adjust size of S^1. When h^0 is small, i.e. the circle small, the black hole looks effectively four-dimensional with electric charges q_I (cf. fig. 6). When h^0 is large, we have a 5D black hole (cf. fig. 7). h^0 is a modulus at infinity, and we find the entropy of an extremal black hole to be moduli independent,

$$S_{4D}(q_0, q_I) = S_{5D}(J = \frac{q_0}{2}, q_I)\,, \tag{2.26}$$

as expected. Both sides can be understood microscopically, but this statement does not survive stringy and quantum corrections.

Five-Dimensional Black Rings

If one consider the harmonic function

$$H^0 = \frac{1}{|\,\vec{x}\,|}, \quad H^I = \frac{p^I}{|\,\vec{x} + R\hat{n}\,|}, \quad H_0 = \frac{q_0}{16}\left(\frac{1}{\vec{x} + R\hat{n}} - \frac{1}{R}\right), \quad H_I = h_I + \frac{\bar{q}_I}{4\,|\,\vec{x} + R\hat{n}\,|}\,, \tag{2.27}$$

one finds black ring solutions in five dimensions. Here $\hat{n}$ is an arbitrary unit vector in R^3. The ring warps around the x^5 direction, which is topologically trivial. R is the ring radius. The ring carries both electric charge and magnetic dipole charges. The latter are analogous to a $F1-$string wrapped on a circle. The near horizon geometry is $AdS_3 \times S^2$, and the entropy reads

$$S = 2\pi\sqrt{l^6 - J^2}, J = \frac{1}{2} + l^3 + P^I \bar{q}^I/2\,. \tag{2.28}$$

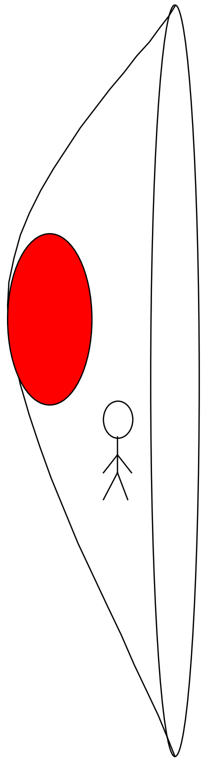

Figure 7: Black Hole in a large Taub-NUT space.

3. Lecture 3: Microscopic Counting of Black String Entropy in Five Dimensional Supergravity

Black String Solutions from Wrapped M5 Branes

Using the construction discussed above we are also able to construct black string solutions by wrapping M5 branes on a four cycle of the compactified $\mathcal{M}_6$. Since the M5 branes are magnetically charged under the background A_4 field, these solutions in the dimensionally reduced theory are magnetically charged black strings. The metric of these backgrounds takes the form

$$ds^2 = \frac{4}{\left(H^3\right)^{1/3}}\left(dtdx^5\right) - \frac{1}{4}\left(H^3\right)^{2/3}\left(dr^2 + r^2 d\Omega_2^2\right)$$

$$F^I = \frac{1}{2}p^I \epsilon_{S^2} \; H^3 = \frac{1}{3!}C_{IJK}H^I H^J H^K, \; H^I = h^I + \frac{p^I}{r} \tag{3.1}$$

The magnetic charge of the solution is given the integral of the $U(1)$ gauge field over the two sphere. However, the above solution has zero horizon area and hence zero entropy.

The near horizon limit of the metric can be obtained by putting $H^I = \frac{p^I}{r}$, $H^3 = \frac{p^3}{r}$, where $p^3 = \frac{1}{3!}C_{IJK}p^Ip^Jp^K$. The metric will become

$$ds^2 = \frac{4r}{(p^3)^{1/3}}\left(dtdx^5\right) - \frac{1}{4r^2}\left(p^3\right)^{2/3}(dr^2 + r^2 d\Omega_2^2)\,,$$

which is a form $AdS_3 \times S^2$. In order to obtain a solution with finite horizon size we have to add momentum q_0 along the black string. This procedure gives

$$ds^2 = \frac{4}{(H^3)^{1/3}}\left(dtdx^5 + H_0(dx^5)^2\right) - \frac{1}{4}\left(H^3\right)^{2/3}\left(dr^2 + r^2 d\Omega_2^2\right)\,.$$

$$F^I = \frac{1}{2}p^I\epsilon_{S^2}\,, \quad H^3 = \frac{1}{3!}C_{IJK}H^IH^JH^K\,, \quad H^I = h^I + \frac{p^I}{r}\,, \quad H_0 = h_0 + \frac{q_0}{16r} \tag{3.2}$$

These solutions are supersymmetric and they have near horizon geometries

$$ds^2 = \frac{4r}{(p^3)^{1/3}}\left(dtdx^5 + \frac{q_0}{16r}(dx^5)^2\right) - \frac{1}{4r^2}\left(p^3\right)^{2/3}(dr^2 + r^2 d\Omega_2^2)\,.$$

which can be factorized into $BTZ \times S^2$. The entropy now can be computed using the Bekenstein Hawking formula (here we use $G_5 = \pi/4$),

$$S = \frac{A}{4G_5} = 2\pi\sqrt{\frac{1}{3!}C_{IJK}p^Ip^Jp^Kq_0}\,. \tag{3.3}$$

Microscopic Counting of Black String Entropy

The entropy of these black strings can be understood microscopically. The authors of [10] studied M5 branes wrapped on generic four cycle and pointed out that these states corresponds to a $1+1$ dimensional CFT with $(0,4)$ supersymmetry. The central charges for the left and right moving sectors of the theory are

$$c_L = C_{IJK}p^Ip^Jp^K + C_{2,I}p^I,, \tag{3.4}$$

$$c_R = C_{IJK}p^Ip^Jp^K + \frac{1}{2}C_{2,I}p^I\,, \tag{3.5}$$

label:central where $C_{2,I}$ are the coefficients of the second Chern class of the $\mathcal{M}_6$. From the gravity point of view, the terms linear in p^I are higher derivative corrections and we will ignore them for the time being. The entropy in a conformal field theory can be related to the central charge through Cardy's formula [9]

$$S = 2\pi\sqrt{\frac{c_L}{6}(L_0 - \frac{c_L}{24})} + 2\pi\sqrt{\frac{c_R}{6}(\tilde{L_0} - \frac{c_R}{24})} \tag{3.6}$$

The formula is valid at asymptotically high temperatures. The black string solutions discussed in the previous section are supersymmetric and thus correspond to having momenta excited in the left moving sector which is the non-supersymmetric side of the CFT. With the identification of $q_0 = L_0 - \frac{c_L}{24}$, we can see an agreement between the microscopic counting of entropy on the CFT side and the Bekenstein Hawking formula.

Suppose we instead excite momenta on the right moving sector of the conformal field theory, we will break all supersymmetry. This corresponds on the gravity side to flipping the sign of the momentum q_0. The black string solutions obtained this way will be non-BPS. However, we can still get agreement between the two ways of accounting of the entropy:

$$S_{grav} = S_{micro} = 2\pi\sqrt{\frac{c_L}{6}|q_0|}\,.$$

More generally, we can excite both the left and the right movers and still get agreement sufficiently near extremality and the entropy formula is

$$S = 2\pi\sqrt{\frac{c_L}{6}h_L} + 2\pi\sqrt{\frac{c_R}{6}h_R} \tag{3.7}$$

$$q_0 = h_L - h_R$$

In fact, agreement persists even when higher derivative corrections are present in the supergravity action (the entropy is not given by the area formula anymore) and the appropriate terms with $c_{2,I}$ are included in expressions for the central charges. The five dimensional black string solutions obtained through dimensional reduction from M theory provide us with a nice example of how entropy associated with black objects can be accounted for microscopically using CFT. It is sometimes argued that black hole entropy counting amounts to counting BPS states and that agreement is achieved because BPS states are protected by supersymmetry in going from weak to strong coupling. The fact that the entropy formulas still match when higher derivative corrections are included even though the solutions are not BPS suggests other mechanisms are at work to ensure the agreement.

One especially interesting special case is when we take $\mathcal{M}_6 = K_3 \times T^2$ and wrap the M5 entirely on the K_3. At weak coupling, this can be thought of as IIA string theory on $K_3 \times S^1$, which is S-dual to the heterotic string on the $T^4 \times S^1$. The K_3 wrapped M5 can be thought of as a fundamental heterotic string on S^1 . With this setup, only one of the p^I is non-vanishing and since C_{IJK} is totally antisymmetric, the leading contribution to the central charges vanish identically leaving only the correction turns terms:

$$c_L = C_{2,I}p^I = 24N\,, \tag{3.8}$$
$$c_R = \frac{1}{2}C_{2,I}p^I = 12N\,, \tag{3.9}$$

where N is the number of M5 branes. This agrees with the world sheet content of heterotic strings. Specifically, in the left moving sector we have 24 physical bosonic degrees of freedom and each of them will contribute an amount of one to the central charge c_L. The right moving sector contains eight bosons and eight fermions. The bosons again carry one unit of central charge each, while the fermions carry one half unit. This makes in total $c_R = 12 = 8 + 4$.

In the supergravity limit the solutions corresponding to the wrapped heterotic strings are singular at the two derivative level, but higher derivative corrections resolve the singularity and yield some geometry.

Near Horizon Physics As we reviewed, the black holes/black strings in question have a near horizon factor AdS_2 or AdS_3, and this fact is essential to our understanding of their microscopic descriptions. For example, it is the basis of the attractor mechanism, as we will see.

Gravity in AdS_3

Here we have excellent control. In fact, if we look at the cases where we have successfully matched the entropy there is always as AdS_3 factor present. This includes the 5D black string discussed above. At a basic level, AdS_3 is a solution of

$$I = \frac{1}{16\pi G}\int d^3x\sqrt{g}(R - \frac{2}{\ell^2}) + I_{bndy}\,,$$

reading

$$ds^2 = (1 + \frac{r^2}{\ell^2})dt^2 + \frac{1}{1 + \frac{r^2}{\ell^2}}dr^2 + r^2 d\phi^2. \tag{3.10}$$

AdS_3 is a homogeneous space with maximal symmetry, and the isometry group is $SL(2,\mathbb{C}) \simeq SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$. The BTZ black hole is a particular quotient of AdS_3 by an element of the isometry group. To see this, consider the AdS_3 metric written as

$$ds^2 = dX_0^2 + dX_1^2 - dX_2^2 - dX_3^2\,, \tag{3.11}$$

embedded as an hyperboloid described by the equation

$$X_0^2 + X_1^2 - X_2^2 - X_3^2 = -\ell^2\,. \tag{3.12}$$

If there is a group element $g \in SL(2,\mathbb{R})$ with $\det g = 1$, we write it as

$$g = \frac{1}{\ell}\begin{pmatrix} X_1 + X_2 & X_3 - X_0 \\ X_3 + X_0 & X_1 - X_2 \end{pmatrix}. \tag{3.13}$$

We can use g to generate the AdS_3 metric by

$$ds^2_{AdS3} = -\ell^2 \mathrm{Tr}(g^{-1}dgg^{-1}dg). \tag{3.14}$$

$SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ acts by conjugation on itself,

$$g \mapsto \rho_L g \rho_R \,,$$

which leaves $\mathrm{d}s^2_{AdS3}$ invariant. However, because of the left (and also right) action of the group on itself $g \mapsto g_L g$, both matrices $\rho_{L,R}$ are only defined up to conjugation, $\rho_L \sim g_L^{-1}\rho_L g_L$. The BTZ black hole is then obtained as a quotient of AdS_3 by some of its isometries, i.e. by identifying

$$g \sim \rho_L g \rho_R \,, \tag{3.15}$$

where both ρ_L and ρ_R are only defined up to conjugation. $SL(2,\mathbb{R})$ has three conjugacy classes labeled by the trace of the matrix g,

- $|\mathrm{Tr}g| > 2$, called hyperbolic,
- $|\mathrm{Tr}g| = 2$, called elliptic, and
- $|\mathrm{Tr}g| < 2$, called parabolic.

These three isometries correspond to, respectively, Lorentz boosts, rotations, and null rotations of Minkowski space. The BTZ black hole is obtained by an identification of AdS_3 under Lorentz boosts, i.e. we choose the hyperbolic conjugacy class. In that class, $\rho_{L,R}$ can be written in simple diagonal form,

$$\rho_L = \begin{pmatrix} e^{2\pi^2 T_+} & 0 \\ 0 & e^{-2\pi^2 T_+} \end{pmatrix}, \quad \rho_R = \begin{pmatrix} e^{2\pi^2 T_-} & 0 \\ 0 & e^{-2\pi^2 T_-} \end{pmatrix}. \tag{3.16}$$

$T_\pm$ are the left- and right-moving temperatures, corresponding to the left- and right-moving sectors of the dual CFT. We now show that the identification (3.15) can be implemented onto a solution of (3.12), if choosing coordinates as

$$X_1 \pm X_2 = \ell\sqrt{\frac{r^2 - r_+^2}{r_+^2 - r_-^2}}\, e^{\pm\pi(T_+u_+ + T_-u_-)}, \quad X_3 \pm X_0 = \ell\sqrt{\frac{r^2 - r_-^2}{r_+^2 - r_-^2}}\, e^{\pm\pi(T_+u_+ - T_-u_-)}. \tag{3.17}$$

Here $r_\pm = \pi\ell(T_+ \pm T_-)$, $u_\pm = \phi \pm t$. The identification (3.15) then corresponds to identifying $\phi \sim \phi + 2\pi$. In terms of the locations of the inner and outer horizons $r_\pm$, the Euclidean solution is

$$ds^2 = f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\phi + i\frac{r_+r_-}{\ell r^2}dt)^2 \tag{3.18}$$

$$f(r) = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{\ell^2 r^2} \tag{3.19}$$

The general rotating BTZ black hole has two horizons, the event horizon r_+ and the inner horizon r_-. Its mass and angular momentum are found to be

$$M = \frac{r_+^2 + r_-^2}{8G\ell^2}\,, \quad J = \frac{r_+ r_-}{2G\ell}\,.$$

The event horizon has Bekenstein-Hawking area law,

$$S = \frac{A_+}{4G} = 2\pi\sqrt{\frac{c}{2}(M\ell + J)} + 2\pi\sqrt{\frac{c}{2}(M\ell - J)}\,. \tag{3.20}$$

The central charge is found according to Brown and Henneaux [11],

$$c = \frac{3\ell}{2G}\,. \tag{3.21}$$

Now in string theory we expect various higher derivative corrections to the action. However, we expect that AdS_3 will still be a solution since it is naturally symmetric. Only the formula for the scale size ℓ will change. Similarly, the BTZ solution will then be a solution since it's just a quotient of AdS_3. On the other hand, the mass, angular momentum and entropy of the black hole will be modified.

The way to handle this is to make use of the full symmetry of the problem. One central point is that quantum gravity on AdS_3 necessarily has the symmetries of a 2D CFT, i.e. the left- and right-moving Virasoro algebras [11]. The best way to show this is to define the stress tensor and show that it has the properties expected. For obtaining the stress tensor of asymptotic AdS_3 on the boundary, we can use the Fefferman-Graham expansion [12] (every order of the expansion is decreased by a factor $e^{\eta/\ell}$)

$$g_{ij} = e^{2\eta/\ell} g_{ij}^{(0)} + g_{ij}^{(2)} + \cdots \tag{3.22}$$

to rewrite the AdS_3 metric, which looks like (for $\eta \to \infty$)

$$ds^2 = d\eta^2 + e^{2\eta/\ell} g_{ij}^{(0)} dx^i dx^j + g_{ij}^{(2)} dx^i dx^j + \cdots\,. \tag{3.23}$$

Here g_{ij}^0 is the conformal boundary metric. The stress tensor is defined in terms of the action as the variation with respect to the conformal boundary metric,

$$\delta I = \frac{1}{2}\int d^2x \sqrt{g}^{(0)} T^{ij} \delta g_{ij}^{(0)}\,. \tag{3.24}$$

In fact, the two-dimensional conformal group describes asymptotic symmetries which are diffeomorphisms leaving $g_{ij}^{(0)}$ invariant. Up to the two derivative level, the relevant action is the Einstein-Hilbert action in three dimensions with cosmological constant,

$$I = \frac{1}{16\pi G}\int d^3x\sqrt{g}\left(R - \frac{2}{\ell^2}\right) - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^2x\sqrt{g}\,\mathrm{Tr}K - \frac{1}{8\pi G\ell}\int_{\partial\mathcal{M}} d^2x\sqrt{g}\,. \tag{3.25}$$

The last two term are needed for a well-defined variational principle (the York-Gibbons-Hawking term) and to make action finite (a boundary counterterm). At the boundary ($\eta \to \infty$) one obtain the AdS_3 boundary stress tensor

$$T_{ij} = \frac{1}{8\pi G\ell}(g^{(2)}_{ij} - (\mathrm{Tr}g^{(2)})g^{(0)}_{ij}) \tag{3.26}$$

by varying the above action and plugging in the Fefferman-Graham expansion. It is also possible to compute the central charge (here $c_L = c_R$ automatically).

The easiest way to obtain central charge is to extract it from the trace anomaly,

$$T^i_i = -\frac{c}{24\pi}R(g^{(0)})\,. \tag{3.27}$$

According to this formula, the action transforms under a Weyl transformation of $g^{(0)}$, $\delta g^{(0)}_{ij} = 2g^{(0)}_{ij}\delta\omega$, as

$$\delta I = \frac{1}{2}\int d^2x\sqrt{g}^{(0)}T^{ij}\delta g^{(0)}_{ij} = \frac{1}{2}\int d^2x\sqrt{g}^{(0)}T^{ij}(2g^{(0)}_{ij}\delta\omega) = -\frac{c}{24\pi}\int d^2x\sqrt{g}^{(0)}R^{(0)}\delta\omega \tag{3.28}$$

If $\delta\omega$ is a constant, and the boundary is S^2, we have

$$\delta I = -\frac{c}{3}\delta\omega\,. \tag{3.29}$$

Now, to compare against the gravity computation, AdS_3 can be written as

$$ds^2 = \ell^2(d\eta^2 + \sinh^2\eta d\Omega_2^2), \tag{3.30}$$

thus

$$g^{(0)}_{ij}dx^i dx^j = \frac{\ell^2}{4}d\Omega_2^2. \tag{3.31}$$

The gravitational action on this background is divergent due to the $\frac{1}{16\pi G}\int d^3x\sqrt{g}(R - \frac{2}{\ell^2})$ term, as $R = \frac{6}{\ell^2}$ on-shell. Putting in a large η cuttoff we find

$$I_{div} = -\frac{\ell}{2G}\eta_{max}\,. \tag{3.32}$$

We would need to add another counter term to cancel this, but this counterterm would then depend explicitly on η_{max}, and thus break diffeomorphism invariance. This contribution is thus the sought-for conformal anomaly. The on-shell action will transform under a diffeomorphism $\eta \to \eta + \delta\eta$, as $\delta I = -\frac{\ell}{2G}\delta\eta$. Thus, from the form of the conformal anomaly (3.29), we see that this is equivalent to a conformal transformation $\delta\omega = \delta\eta$, and thus we can read off the central charge

$$c = \frac{3\ell}{2G}\,. \tag{3.33}$$

This is the Brown-Henneaux central charge formula [11], which they obtained by a different route, namely by identifying asymptotic symmetries in the Hamiltonian formulation of 2+1-dimensional gravity. The advantage of the present derivation is that it generalizes to arbitrary theories including higher derivatives, where the constraint analysis needed for the Brown-Henneaux analysis might not be straight-forward.

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