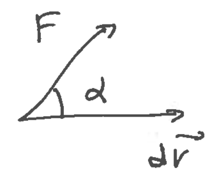
Section B: Work and Energy

Definition of work:

$$dU = \vec{F}\cdot d\vec{r} = F\,dr\cos\alpha = F_x\,dx + F_y\,dy + F_z\,dz$$

i.e: displacement x component of force in direction of motion.



The other type of components (e.g support forces) do no work and can be disregarded.

Units of work: Newton x meter = Joule.

The total work between points $1 \rightarrow 2$ is given by the integral

$$U_{1\rightarrow 2} = \int_1^2 \vec{F}\cdot d\vec{r}$$

Work & kinetic energy

Once we know the work done on a particle, we can translate that into change in its kinetic energy $T = \frac{1}{2} m v^2$.

That is $\boxed{U_{1\to 2} = T_2 - T_1}$ work-energy equation

Total work done by all forces translates into change in kinetic energy, as above.

proof: $U_{1\to 2} = \int_1^2 \vec{F} \cdot d\vec{r}$

$\vec{F} = m\vec{a} \implies U_{1\to 2} = \int_1^2 m\vec{a} \cdot d\vec{r}$

$a = \frac{d\vec{v}}{dt} \qquad d\vec{r} = \vec{v}\, dt$

$$U_{1\to 2} = m \int_1^2 \left(\vec{v} \cdot \frac{d\vec{v}}{dt}\right) dt = \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2 = T_2 - T_1$$

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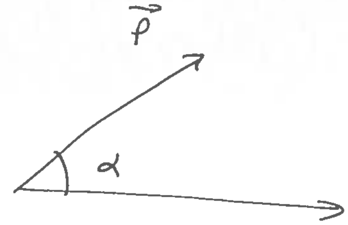
Example 1: constant force



$$U_{1\to 2} = \int_1^2 \vec{F}\cdot d\vec{r} = \int_1^2 |F| \cos\alpha \, dx = |F| \cos\alpha \, (x_2 - x_1)$$

Example 2: elastic force

In one direction $F = -kx$

x = distance from unstretched position.

$$\int_1^2 \vec{F}\cdot d\vec{r} = -\int_{x_1}^{x_2} kx \, dx = \frac{1}{2}\left(x_1^2 - x_2^2\right)$$

This can be either positive or negative.

Example 3: gravity

On earth $F = -mg$, so this is an example of constant force.

$$U_{1\to 2} = -mg \int_1^2 dy = -mg\,(y_2 - y_1)$$

y = height.

But for gravity not necessarily close to the earth surface

$$\vec{F} = -G \frac{Mm}{r^2} \hat{r}$$

For radial motion:

$$U_{1\to 2} = \int_1^2 \vec{F} \cdot d\vec{r} = -\int_1^2 \frac{GMm}{r^2} dr = -GmM \left(\frac{1}{r_2} - \frac{1}{r_1} \right).$$

<u>Advantages of using</u> work-energy

If we need to only know final velocity, we can use the work-energy relation. This offers the simplifications:

- Can only consider forces that do work.
- Can ignore internal forces (see example below).
- Can discuss initial and final state only, without all the details of what happens in the interim.

Power: amount of work done per unit time:

$$dU = \vec{F}\, d\vec{r}$$

$$P \equiv \frac{dU}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt} = \vec{F} \cdot \vec{v}$$

Units of power are $\frac{\text{Joule}}{\text{sec}} = \text{Watt}$.

Sample problem 3.15

Satelite of mass m is in an elliptical orbit around the earth. At distance $h_1 = 500$ km it has $V_1 = 30{,}000 \frac{km}{hr}$.

Determine V_2 when the height is $h_2 = 1200$ km.

$$U_{1\to 2} = -mgR^2\left(\frac{1}{r_2} - \frac{1}{r_1}\right) \qquad \text{using } GM_{earth} = gR_{earth}^2$$

$$= \frac{1}{2} m V_2^2 - \frac{1}{2} m V_1^2$$

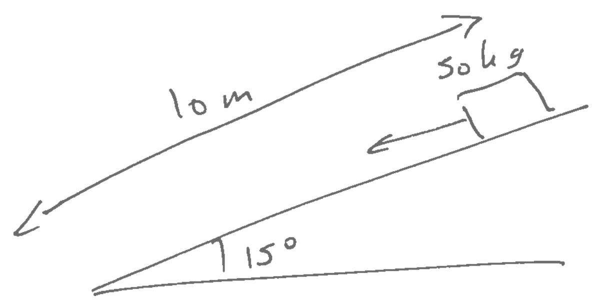
$$V_2^2 = V_1^2 + 2gR^2\left(\frac{1}{r_2} - \frac{1}{r_1}\right).$$

$R_{earth} = 6.371 \cdot 10^3$ km , so $\quad r_2 = h_2 + R_{earth}$

$r_1 = h_1 + R_{earth}$

giving $\quad V_2 = 27,590 \frac{km}{hr}$ (slower at larger distance for the earth!)

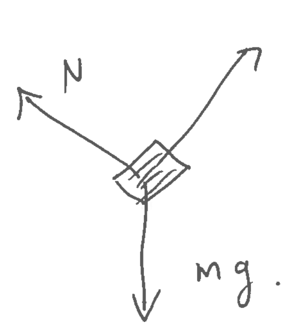
Sample problem 9.11



$V_{in} = 4 \frac{m}{sec}$

Calculate final velocity when crate reaches bottom. Coeff. of kinetic friction is 0.3.

Free body diagram



Work done by gravity

$50 \cdot 9.8 \cdot (10 \sin 15°)$.

Work done by friction: for that we need to figure out the normal force

$$N = 50 \cdot 9.81 \cdot \cos 15^\circ = 473.$$

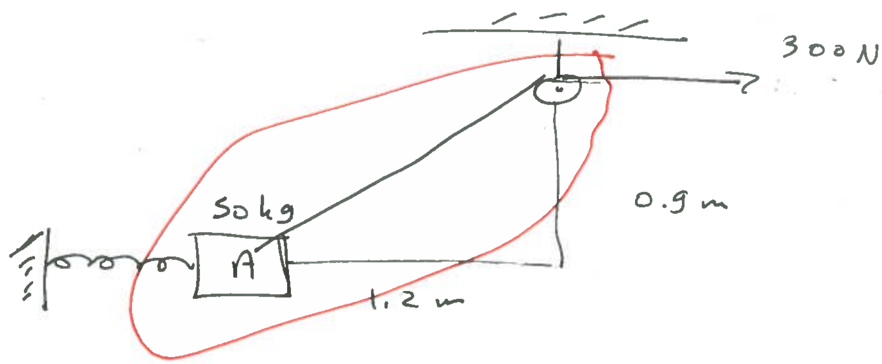
$$\text{Friction} = N \cdot 0.3 = 142 \text{ Newton.}$$

Work done by friction $= -142 \cdot 10\,m$ (−) sign because it opposes the motion

$$\text{Total work} = -151.9 \text{ Joule} = \tfrac{1}{2} m V_2^2 - \tfrac{1}{2} m V_{in}^2$$

$$\Longrightarrow \quad V_2 = 3.15 \ \frac{m}{sec}$$

Sample problem 3.13 This illustrates that due to action-reaction, internal forces can be ignored. The work each component of the system does on other components is exactly cancelled by the work done on it.



The system we are considering is encircled. The force the pulley and A exert on each other is equal and opposite in direction, so that part of the work cancels. We need to draw active-body diagram, taking into account only forces external to our system.

When we do that we see two forces:

- spring connected to the left.
- cable connected to the right

* Work done by cable: $F \cdot \Delta X$ (constant force).

ΔX = amount of cable moving in horizontal direction =

$$\underbrace{\sqrt{1.2^2 + 0.9^2}}_{\text{(at position 1)}} - \underbrace{0.9}_{\text{(at position 2)}} = 0.6\ m.$$

$$0.6\ m \times 300\ N = 180\ \text{Joule}.$$

* work done by spring: we are given spring unextended length implicitely since at position 1 the extension is $0.233\ m$.

The stiffness is $80\ \frac{N}{m}$

$$\text{work done} = -\frac{1}{2} k (x_2^2 - x_1^2) = \frac{1}{8} \cdot 80 \left[(0.233)^2 - (0.233 + 1.2)^2 \right]$$
$$= -80\ \text{Joule}.$$

$$\text{Total work} = 100\ J = \frac{1}{2}\, 50\, (V_2^2 - V_1^2) = \frac{1}{2} \cdot 50\, V_2^2 \quad (V_1 = 0)$$

$$\Rightarrow \boxed{V_2 = 2\ \frac{m}{sec}}$$

Potential energy:

For some forces, one can write the work done in terms of potential energy differences between initial & final states. This leads to a new energy-work relation.

Example: gravity:

write $V_g = mgh$ = work done to get to height h from height 0 (against gravity).

Then the work done by gravity is:

$$U_{1\to2} = -\Delta V_g \qquad \text{where } \Delta V_g = mg(h_2 - h_1)$$
$$= mg\Delta h$$

(−) sign means loss of potential energy translates to work done = gain in kinetic energy.

For large altitude changes:

$$F = G\frac{M_e \cdot m}{r^2} = mg \cdot \frac{R_e^2}{r^2}$$

$$\int_{r_1}^{r_2} mgR^2 \cdot \frac{dr}{r^2} = -mgR^2\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = (V_g)_2 - V_g$$

$$V_g = -\frac{mgR_e^2}{r}$$

$$U_{1\to 2} = \Delta V_g = -mgR^2\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

Elastic potential energy:

$$V_e = \int_0^x kx\,dx = \frac{1}{2}kx^2$$

$$\Delta V_e = \frac{1}{2}k\left(x_2^2 - x_1^2\right) = -U_{1\to 2}$$

work done by spring = (−) change in it's potential energy.

Work energy relation:

$\Delta T = U_{1\to 2}$; now we can write some of $U_{1\to 2}$, for the conservative forces, as $-\Delta V$, minus the change in their potential energy. This gives:

$$\Delta T + \Delta V = U'_{1\to 2} \qquad \text{(sum over all non-conservative forces)}.$$

or $$(T_1 + V_1) + U'_{1\to 2} = (T_2 + V_2).$$

if $U'_{1\to 2} = 0$, then $\Delta T + \Delta V = 0$, the total energy $T+V$ is conserved.

Forces from potential energy:

For conservative forces:

$$U_{1\to 2} = \int_1^2 \vec{F}\cdot d\vec{r} = -\Delta V = -(V_2 - V_1)$$

This means that the term $\vec{F}\, d\vec{r}$ is a total derivative $\vec{F} \cdot d\vec{r} = -dV$.

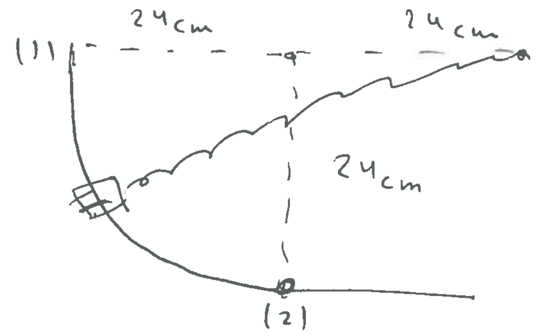
$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz.$$

$$\vec{F} \cdot d\vec{r} = F_x\, dx + F_y\, dy + F_z\, dz.$$

$$\longrightarrow F_x = -\frac{\partial V}{\partial x} \qquad F_y = -\frac{\partial V}{\partial y} \qquad F_z = -\frac{\partial V}{\partial z}$$

or $\vec{F} = -\vec{\nabla} V$ (gradient)

Sample problem 3.16



6 kg slider released from rest (positn 1) and slides to the bottom of a circular arc of radius 24cm. The unstretched length of the spring is 24 cm, and $k = 200 \frac{N}{m}$. What is the velocity in the position (2)?

Gravitational potential energy:

$$V_1 = 0$$

$$V_2 = mg(h) = mg \cdot (-0.24) = -14.1 \text{ Joule.}$$

Spring potential energy:

$$V_1 = \frac{1}{2} k x_1^2 = \frac{1}{2} \cdot 200 \cdot 0.24^2 = 5.76 \text{ Joule.}$$

$$V_2 = \frac{1}{2} k x_2^2 = \frac{1}{2} 200 \left(0.24\sqrt{2} - 0.24\right)^2 = 0.988 \text{ Joule.}$$

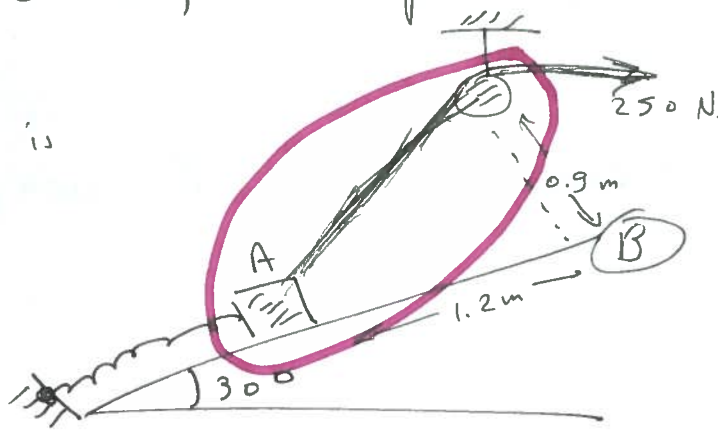
$$T_1 + V_1 = T_2 + V_2 \quad \text{(energy is conserved).}$$

$$0 + 5.76 + 0 = \frac{1}{2} m V^2 - 14.1 + 0.988.$$

$$\Longrightarrow V_2 = 2.5 \cdot \frac{m}{sec}.$$

Sample problem 3.17

10 kg slider moves on a guide along an inclined plane. The attached spring has $k = 60 \frac{N}{m}$, and in position A it is stretched 0.6 m. The slider is initially at rest, what is the speed at position B?



looking at the system encircled, we only have to consider the external forces: gravity + spring (conservative) and cable (not).

The non-conservative force does work.

$$U_{A \to B} = 250 \cdot 0.6 = 150 \text{ J}$$

0.6 = difference in cable length $= 1.5 - 0.9$.

Gravity potential energy

$$V_A = 0$$

$$V_B = mgh = 10 \cdot 9.8 \, (1.2 \sin 30^\circ) = 58.9 \text{ J}.$$

elastic potential energy:

$$V_A = \tfrac{1}{2} k x_A^2 = \tfrac{1}{2} (60) \cdot 0.6^2 = 10.8 \text{ J}.$$

$$V_B = \tfrac{1}{2} k x_B^2 = \tfrac{1}{2} (60) \cdot (0.6 + 1.2)^2 = 97.2 \text{ J}.$$

$$T_A + V_A + U'_{A \to B} = T_B + V_B.$$

$$0 + 0 + 10.8 + 150 = \tfrac{1}{2} \cdot 10 \cdot v_B^2 + 58.9 + 97.2.$$

$$v_B = 0.974 \ \tfrac{m}{sec}.$$