Summary 3

January 26, 2015

Abstract

This is a summary of the classes given in the second half of January, covering roughly sections 2.3, 2.4 and 2.6 in the textbook.

Learning Goals

Vectors describing motion on the plane: location, velocity and acceleration.

Coordinate system and the axes (unit vectors) associated with them.

Rectangular coordinates.

Polar coordinates.

Transformation between coordinate systems.

Motion on a plane

Location on a plane is described by a vector $vecr$. When we describe the plane in terms of coordinates this vector corresponds to two numbers instead of one. But for now we don't commit to a specific set of coordinates.

The vector $\vec{r}$ describes the distance between the location of a point and a fixed origin (draw). This includes information about both magnitude (distance) and direction of the point relative to the fixed origin.

When particle moves in time, over a small increment its position changes from $\vec{r}$ to $\vec{r}+\Delta\vec{r}$. The *displacement* $\Delta\vec{r}$ is independent of the arbitrary choice of origin.

The displacement on the plane is very much like the displacement on a line, except it has two independent pieces of information: magnitude and direction. This is shown in figure 2.5, where the magnitude of the displacement vector is called simply the distance.

Velocity

As before, we can define the average velocity during the time Δt, that is $\vec{v}_{average} = \frac{\Delta \vec{r}}{\Delta t}$. This is a vector, it has magnitude and direction. Similarly, the velocity is defined as the time increment Δt approaches zero.

The velocity has magnitude, called the speed, and direction – the tangent to the curve described by the particle motion (draw). it is a time derivative of the displacement vector. If $\vec{r}(t)$ describes the displacement of the particle as function of time, then

$$\vec{v} = \frac{d\vec{r}}{dt} \tag{1}$$

Common confusion: the speed, magnitude of $\vec{v}$, is not the same as the time derivative of the distance from the origin $r = |\vec{r}|$. For example, consider circular motion where there is certainly a velocity, but the distance from the origin does not change at all.

Acceleration

Now that we understand that vector can change with time in two ways, magnitude and direction, we can define the time derivative of the velocity: acceleration. This is simply

$$\vec{a} = \frac{d\vec{v}}{dt} \tag{2}$$

Unlike the velocity, the direction of the acceleration is not easy to visualize. It roughly tells you how *curved* the path of the particle is.

Rectangular coordinates

Given a coordinate system, a vector is specified by two numbers. Easiest to visualize is the rectangular coordinate system, where the displacement vector can be written as$\vec{r} = (x, y)$. Or more formally $\vec{r} = x\hat{i} + y\hat{j}$. The units vectors in the x, j directions are also called $\hat{e}_x$ or $\hat{x}$, etc.

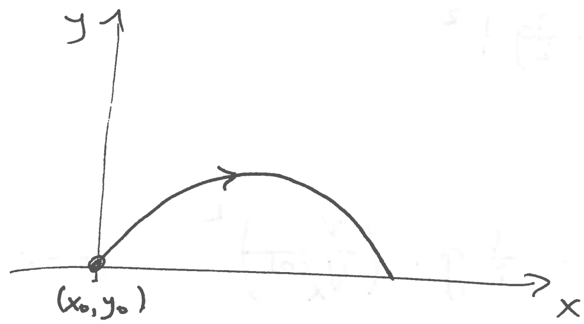
Once we have the coordinates of the path as function of time $(x(t), y(t))$, we can treat both functions independently. So for example the velocity is obtained by differentiating in each direction separately, $\vec{v} = (\dot{x}(t), \dot{y}(t))$. Same goes for the acceleration. So we can solve any problem we discussed for rectilinear motion by solving it separately for the x and y motion.

The magnitude and direction of vector can be found from their coordinates. For example if $\vec{v} = (v_x, v_y)$, then

$$|\vec{v}| = \sqrt{v_x^2 + v_y^2} \qquad tan(\theta) = \frac{v_y}{v_x} \tag{3}$$

This is demonstrated in figure (2.7).

Projectile motion: example of plane motion.



$a_x = 0$

$a_y = -g$

(with assumption that acceleration due to gravity is constant)

$\frac{dv_x}{dt} = 0 \qquad V_x = V_x^{(0)}$

$\frac{dv_y}{dt} = -g \qquad V_y = -gt + V_y^{(0)}$

$V_x^{(0)}, V_y^{(0)}$ initial velocities.

[describe gravity, define g etc.]

integrate again:

$V_x = \frac{dx}{dt} \rightarrow \qquad X = X_0 + V_x^{(0)} \cdot t$

$V_y = -gt + V_y^{(0)} \rightarrow \qquad y = y_0 + V_y^{(0)} t - \frac{1}{2} g t^2$

(x_0, y_0) initial position, can be chosen as $(0,0)$.

To get the shape of motion eliminate t:

$$x = V_x^{(0)} t \longrightarrow \qquad t = \frac{x}{V_x^{(0)}}$$

$$y = V_y^{(0)} t - \frac{1}{2} g t^2 =$$

$$= \frac{V_y^{(0)}}{V_x^{(0)}} x - \frac{1}{2} g \cdot \left(\frac{x}{V_x^{(0)}} \right)^2 \longrightarrow \text{parabola}$$

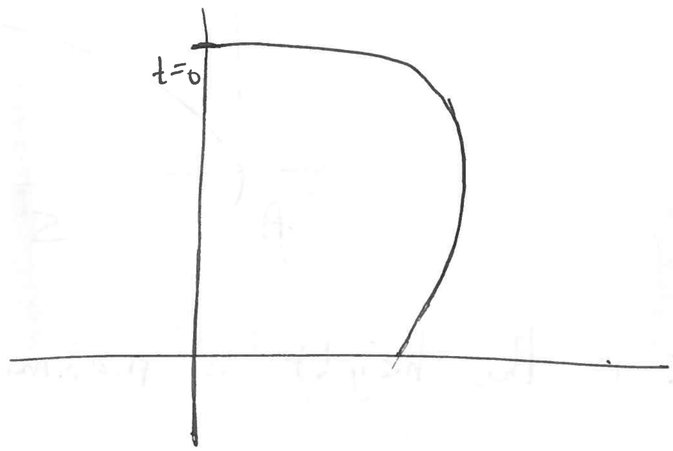
Sample problem 2.5

$V_x = 50 - 16t \ \frac{m}{sec}$ $X = 0$ when $t = 0$

$y = 100 - 4t^2 \ \underline{m}$.

Plot the path of particle, determine velocity & acceleration when $y = 0$.

[parametric vs. shape]



$$X = \int V_x dt = \int_0^t (50 - 16t')\,dt = 50t - 8t^2$$

$$V_y = -8t \ \frac{m}{sec}$$

$$a_x = \frac{dV_x}{dt} = -16 \ \frac{m}{sec^2} \qquad a_y = -8 \ \frac{m}{sec^2}$$

when $y = 0$ $t^2 = 25$

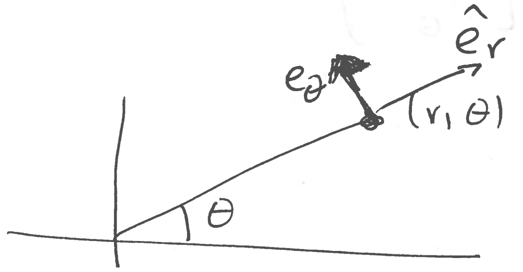
$t = 5$

Subs $\rightarrow$ $V_x = -30$ $V_y = -40$. $|v| = 50 \ \frac{m}{sec}$.

$|a| = 17.89 \ \frac{m}{sec^2}$

Polar coordinates:

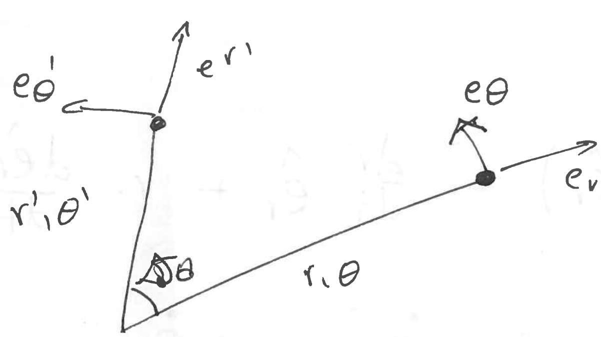
Choose origin and an axis, describe point by distance and angle.



As before, we can define unit vectors $\hat{e}_r, \hat{e}_\theta$ which have length 1 and point in the direction of increasing r, θ.

In terms of those quantities $\vec{r} = r \cdot \hat{e}_r$.

Time derivatives: Here the unit vectors also change when location changes slightly



units vectors rotate by angle $\Delta\theta$

The rotat ~ω : (derive for any set of coordinate axes)

$$\begin{bmatrix} \hat{e}_r' \\ \hat{e}_\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \end{bmatrix}$$

expand for small θ

$$\delta e_r = +(\delta\theta)\,\hat{e}_\theta$$

$$\delta e_\theta = -(\delta\theta)\,e_r$$

$$\Longrightarrow \begin{bmatrix} \dot{e}_r = \dot{\theta}\,\hat{e}_\theta \\ \dot{e}_\theta = -\dot{\theta}\,\hat{e}_r \end{bmatrix}$$

which means:

$$\vec{v} = \frac{d}{dt}\vec{r} = \frac{d}{dt}(r\hat{e}_r) = \frac{dr}{dt}\hat{e}_r + r\cdot\frac{d\hat{e}_r}{dt}$$

$$= \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$$

$|v| = \sqrt{v_r^2 + v_\theta^2}$ with $v_r = \dot{r}$ $\quad v_\theta = r\dot{\theta}$

Acceleration:

Similarly

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta\right) =$$

$$= \ddot{r}\,\hat{e}_r + \dot{r}\,\dot{\hat{e}}_r + \dot{r}\dot{\theta}\,\hat{e}_\theta + r\ddot{\theta}\,\hat{e}_\theta + r\dot{\theta}\,\dot{\hat{e}}_\theta$$

$$= \underbrace{\left(\ddot{r} - r\dot{\theta}^2\right)}_{a_r}\hat{e}_r + \underbrace{\left(r\ddot{\theta} - 2\dot{r}\dot{\theta}\right)}_{a_\theta = \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)}\hat{e}_\theta$$

Circular motion: if $r = R =$ constant.

$$\dot{r} = \ddot{r} = 0$$

$V_r = 0$ $\qquad$ $V_\theta = r\hat{\theta}$

$a_r = -r\dot{\theta}^2$ $\qquad$ $a_\theta = r\ddot{\theta}$

next: sample problem 2.9 (example of constrained motion)

Changing coordinates:

Given (r, θ) then $x = r\cos\theta$

$y = r\sin\theta$

Given x, y then $r = \sqrt{x^2+y^2}$

$\tan\theta = \frac{y}{x}$.