Central force motion:

Consider the motion of small body around a more massive one, earth around the sun for example. The sun does not move much, and the distance is so large the sun and the earth can be both treated as points. The force between them is

$$F_r = -G\frac{Mm}{r^2}$$

$$F_\theta = 0$$

This is called central force, always pointing in the radial direction. Predicting the motion of the planets is one of the first applications of Newton's laws.

$$F_\theta = 0 \implies m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0$$

$$\implies \frac{d}{dt}(r^2\dot{\theta}) = 0$$

$r^2\dot{\theta} = h =$ Constant. This is the conservation of angular momentum, true for any central force. (since $v = r\dot{\theta}$, and $\vec{v}, \vec{r}$ perpendicular $|\vec{r} \times \vec{v}| = r^2\dot{\theta}$).

The radial equation is then:

$$m(\ddot{r} - r\dot{\theta}^2) = F_r = -G\frac{Mm}{r^2}$$

We can make the following simplifications:

- Instead of solving for $r(t)$, solve for $r(\theta) \rightarrow$ This gives the shape of the orbit, not how the motion proceeds as function of time.
- Solve for $u = \frac{1}{r}$

This gives:

$$r = \frac{1}{u} \longrightarrow \dot{r} = -\frac{\dot{u}}{u^2}$$

$$r^2\dot{\theta} = h \longrightarrow \dot{\theta} = hu^2 \longrightarrow \dot{r} = -h\frac{\dot{u}}{\dot{\theta}} = -h\frac{du}{d\theta}$$

$$\dot{r} = -h\frac{du}{d\theta} \longrightarrow \ddot{r} = -h\frac{d^2u}{d\theta^2}\cdot\dot{\theta} = -h^2u^2\frac{d^2u}{d\theta^2}$$

With those expressions for $\dot{r}, \ddot{r}$ we get:

$$\ddot{r} - r\dot{\theta}^2 = -h^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u}(hu^2)^2 = -h^2u^2\frac{d^2u}{d\theta^2} - h^2u^3$$

$$= -\frac{GM}{r^2} = -GMu^2$$

$$h^2\frac{d^2u}{d\theta^2} + h^2u = +GM$$

$$\boxed{\frac{d^2u}{d\theta^2} + u = \frac{GM}{h^2}}$$

This is like an equation for harmonic oscillator.

The solution is:

$$\boxed{u = C\cdot\cos(\theta) + \frac{GM}{h^2}}$$

where C is an arbitrary constant

We can interpret the various trajectories, for different values of C, in terms of their shape and their energy. Starting with the energy:

$$E = K + V = \frac{1}{2} m \vec{v}^{\,2} - G \frac{mM}{r^2}$$

$$= \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) - G \frac{mM}{r}$$

This is a constant since the force is conservative.

Evaluate the constant for the above solution, for $\theta = 0$ where $\dot{u} = 0$ and hence $\dot{r} = 0$, and

$$u = C + \frac{GM}{h^2}$$

$$E = \frac{1}{2} m \cdot \frac{1}{u^2} (h u^2)^2 - G m M u$$

$$\frac{2E}{m} = h^2 u^2 - 2GM \cdot u =$$

$$= h^2 C^2 + \cancel{2GMC} + \frac{G^2 M^2}{h^2} - \cancel{2GMC} - \frac{2G^2M^2}{h^2}$$

$$= h^2 C^2 - \frac{G^2 M^2}{h^2}$$

So the energy tells you about the constant C:

$$E > 0 \longrightarrow C > \frac{GM}{h^2}$$

$$E < 0 \longrightarrow C < \frac{GM}{h^2}$$

$$E = 0 \longrightarrow C = \frac{GM}{h^2}$$

The constant C also determines the <u>shape</u> of the trajectory. The equation

$$U = C\cos(\theta) + \frac{GM}{h^2}$$

defines a <u>conic section</u>. To see the shape trade C for e, the eccentricity of the orbit. Define

$$d = \frac{1}{C}$$

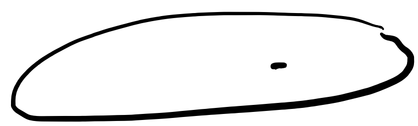
$$ed = \frac{h^2}{GM} \longrightarrow e = \frac{h^2 C}{GM}$$

Then:

$$\frac{1}{r} = \frac{1}{d}\cos(\theta) + \frac{1}{ed}$$

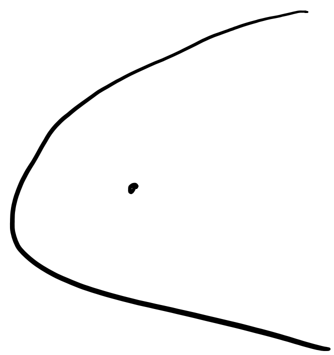
$e<1$: ellipse, r is minimal when $\theta=0$ and maximal when $\theta=\pi$

(This has $E<0$)



$e>0$: hyperbola

(This has $E>0$)



There are two values of θ for which $\frac{1}{r}=0$

$e=0$: parabola, still unbounded orbit.
