PHYS 216 Assignment 9 **Due: 22 March 2017**

1. [15] Show that for a one-dimensional collision between two objects, the change in kinetic energy Q is related to the coefficient of restitution e by

$$Q = \frac{1}{2}\mu v^2(1 - e^2),$$

where μ is the reduced mass and v is the initial difference in the velocities of the two objects.

Answer:

The coefficient of restitution is defined by

$$e = \frac{|v_1' - v_2'|}{|v_1 - v_2|},$$

so $e^2 = (v_1' - v_2')^2/v^2$. Q is defined by

$$Q = T - T' = \frac{1}{2m_1}p_1^2 + \frac{1}{2m_2}p_2^2 - \frac{1}{2m_1}p_1'^2 - \frac{1}{2m_2}p_2'^2.$$

It is simplest to work in the centre-of-mass frame. In this frame the total momentum is zero, so $p_1^2 = p_2^2$ and $p_1'^2 = p_2'^2$. Therefore

$$Q = \frac{1}{2\mu}(p_1^2 - p_1'^2),$$

where $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass.

Since the total momentum is zero, $m_1 v_1 + m_2 v_2 = 0$ and $m_1 v_1' + m_2 v_2' = 0$, so

$$v^2 = (v_1 - v_2)^2 = \left(v_1 + \frac{m_1}{m_2}v_1\right)^2 = \frac{p_1^2}{\mu^2}.$$

Similarly,

$$e^2 v^2 = (v_1' - v_2')^2 = \frac{p_1'^2}{\mu^2}.$$

Substituting these two results into the equation for Q give the expected answer,

$$Q = \frac{\mu^2}{2\mu}(v^2 - v^2 e^2) = \frac{1}{2}\mu v^2(1 - e^2).$$

2. A rocket, whose total mass is m_0, contains a mass of fuel ϵm_0, where $0 \leqslant \epsilon \leqslant 1$. Suppose that the the fuel is burned at a constant rate of k kg/s and that the relative exhaust velocity V is constant.

   (a) [10] If the rocket starts from rest and there are no external forces, what distance will it have travelled at the time that it runs out of fuel?

   (b) [5] What is the maximum possible value of this distance?

Answers:

(a) The problem is one-dimensional, so the equation of motion for the rocket, while it is burning fuel, is

$$m\frac{dv}{dt} = -\frac{dm}{dt}V.$$

(We need the minus sign on the right side because dm/dt is negative.) Note that the mass m on the left side of the equation is *not* a constant. To solve the equation multiply by dt and separate the variables v and m,

$$dv = V\frac{dm}{m}.$$

This is easily integrated,

$$\int_0^v dv = -V\int_{m_0}^{m}\frac{dm}{m}$$

which gives

$$v(m) = -V\ln\left(\frac{m}{m_0}\right).$$

Since the burn rate is constant, $m = m_0 - kt$. Therefore

$$v(t) = -V\ln\left(\frac{m_0 - kt}{m_0}\right) = -V\ln\left(1 - \frac{kt}{m_0}\right).$$

To find the distance travelled we need to integrate the velocity,

$$v(t) = \frac{dx}{dt} = -V\ln\left(1 - \frac{kt}{m_0}\right).$$

so

$$x = -V\int\ln\left(\frac{m_0 - kt}{m_0}\right)dt = -V\int\ln\left(1 - \frac{kt}{m_0}\right)dt.$$

To integrate this let $u = 1 - kt/m_0$ and use the fact that the integral of $\ln(u)$ is $u\ln(u) - u$. Note that the initial value of u is 1.

$$\begin{aligned} x &= \frac{m_0 V}{k}\int_1^u \ln(u)du = \frac{m_0 V}{k}[u\ln(u) - u + 1]. \\ &= \frac{m_0 V}{k}\left[\left(1 - \frac{kt}{m_0}\right)\ln\left(1 - \frac{kt}{m_0}\right) + \frac{kt}{m_0}\right] \end{aligned}$$

The time required to burn all the fuel is $t = \epsilon m_0/k$ so the distance traveled at the time that the fuel is exhausted is

$$x = \frac{m_0 V}{k}\left[(1-\epsilon)\ln(1-\epsilon) + \epsilon\right]$$

(b) This function is zero if $\epsilon = 0$ and increases monotonically as one increases ϵ [to see this note that $dx/d\epsilon = -(m_0 V/k)\ln(1-\epsilon)$]. Therefore the distance is maximized by setting $\epsilon = 1$. In that case, the rocket consists entirely of fuel, and the maximum distance is $m_0 V/k$. Carrying a lot of fuel, burning it slowly, and with a high exhaust velocity, maximizes the distance travelled.