PHYS 216 Assignment 8 **Due: 15 March 2017**

1. (a) [5] Show that the radius for a circular orbit of a geosynchronous (24 hour period) Earth satellite is about 6.6 Earth radii.

   (b) [5] The distance to the Moon is about 60.3 Earth radii. From this calculate the sidereal month (the period of the Moon's orbit) in days.

   Answers:

   (a) The period of a circular orbit of radius r is given by Kepler's third law,

   $$T^2 = \frac{4\pi^2 r^3}{GM_\oplus}$$

   Solving for r and substituting $T = T_{\rm geo}24$ hrs and $M_\oplus = 6.97 \times 10^{24}$ kg we find

   $$r_{\rm geo} = \left(\frac{T^2 GM_\oplus}{4\pi^2}\right)^{1/3} = \left(\frac{(24 \times 3600)^2 (6.67 \times 10^{-11} \times 6.97 \times 10^{24}}{4\pi^2}\right)^{1/3} = 42240 \text{ km} = 6.63 R_\oplus.$$

   (b) Since $T^2 \propto r^3$,

   $$\left(\frac{T_{\leftmoon}}{T_{\rm geo}}\right)^2 = \left(\frac{r_{\leftmoon}}{r_{\rm geo}}\right)^3,$$

   so

   $$T_{\leftmoon} = T_{\rm geo}\left(\frac{r_{\leftmoon}}{r_{\rm geo}}\right)^{3/2} = 1 \text{ day} \times \left(\frac{60.3}{6.63}\right)^{3/2} = 27.43 \text{ days}.$$

2. [10] Show that the period of a satellite in a circular orbit just above the surface of the Earth is the same as the period of oscillation of a particle dropped into a hole drilled through the Earth. (Assume that the density of the Earth is constant.)

   Answer:

   Again we use Kepler's third law, with $r = R_\oplus$

   $$T = 2\pi\sqrt{\frac{R_\oplus^3}{GM_\oplus}}.$$

   According to Newton's theorems, a particle located at radius r inside a hole drilled through the Earth feels a force equal to the gravitational attraction of the mass interior to r. Mass outside a sphere of radius r has no net gravitational force on the particle. The mass inside r is $M(r) = 4\pi r^3 \rho/3$, where ρ is the mean density of the Earth. The total mass of the Earth is $M_\oplus = 4\pi\,{}^{3}_{\oplus}\rho/3$. Dividing these two results gives

   $$M(r) = \left(\frac{r}{R_\oplus}\right)^3 M_\oplus.$$

   Therefore the force on the particle is

   $$F = -\frac{GM(r)m}{r^2} = -\frac{GM_\oplus m r}{R_\oplus^3} = -kr$$

where $k = GM_\oplus m/R_\oplus^3$. This is the same as the force law for a simple harmonic oscillator with frequency $\omega = \sqrt{k/m}$. The period of the oscillation is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{2\pi m}{k}} = 2\pi\sqrt{\frac{R_\oplus^3}{GM_\oplus}}.$$

3. [10] An artillery shell is fired at an angle of $\theta_0 = 30°$ from the vertical direction, with initial speed v_0. At the highest point in its trajectory the shell bursts into two equal fragments, one traveling directly upward at speed $v_0/2$. What is the direction and speed of the other fragment? (You may neglect air resitance.)

   Answer:

   During the shell's travel, there is no force acting in the horizontal direction, so the horizontal component of velocity $v_0 \sin\theta_0 = v_0/2$ is conserved. At the top of the trajectory the vertical component of velocity v_z is zero. Therefore, the momentum of the shell just before it bursts is

$$\boldsymbol{p}_{\text{initial}} = \frac{mv_0}{2}.\boldsymbol{i},$$

   where m is the mass of the shell.

   The burst of the shell is not due to external forces, so the total momentum just after the burst is the same as the momentum just before the burst, $\boldsymbol{p}_{\text{final}} = \boldsymbol{p}_{\text{initial}}$. The final momentum is the sum of that of the two fragments. Let the second fragment have velocity components v_x, v_y and v_z. Then

$$\boldsymbol{p}_{\text{final}} = \frac{m}{2}\frac{v_0}{2}\boldsymbol{k} + \frac{m}{2}v_x\boldsymbol{i} + \frac{m}{2}v_y\boldsymbol{j} + \frac{m}{2}v_z\boldsymbol{k}.$$

   This must be equal to $\boldsymbol{p}_{\text{initial}} = (mv_0/2)\boldsymbol{i}$, which means that all components must be equal. Therefore,

$$\begin{aligned} v_x &= v_0, \\ v_y &= 0, \\ v_z &= -v_0/2, \end{aligned}$$

   The speed of this fragment is $\sqrt{v_0^2 + (v_0/2)^2} = \sqrt{5}v_0/2$. Its direction is $\arctan(1/2) = 26.56°$ downward from the horizontal, in the forward direction.