PHYS 216 Assignment 7 **Due: 8 March 2017**

1. [10] If the solar system were embedded in a uniform cloud of dark matter (matter that does not interact other than gravitationally) ρ, show that the force on a planet of mass m at a distance r from the Sun would be given by

$$F(r) = -\frac{GM_\odot m}{r^2} - \frac{4\pi}{3}G\rho m r.$$

Answer:

To solve this we have to assume that the cloud is spherically symmetric about the Sun (or infinite in extent). Then, by Newton's theorems, the force on the planet is the same as the force that would arise if all the mass interior to the planet were concentrated at the centre. Dark matter at a distance greater than r would exert no net force.

The total mass interior to r is

$$M(r) = M_\odot + \frac{4\pi r^3}{3}\rho,$$

so the force will be

$$F(r) = -\frac{GM(r)m}{r^2} = -\frac{GM_\odot m}{r^2} - \frac{4\pi}{3}G\rho m r.$$

2. [20] A satellite is placed into a low-lying orbit by launching it with a two-stage rocket from Cape Canaveral with speed v_0 inclined from the vertical by an angle θ_0. Upon reaching apogee (highest point) of the initial orbit, the second stage is ignited, generating a velocity boost Δv_1 that places the payload into a circular orbit (see the figure below).

   (a) Calculate the additional speed boost Δv_1 required of the second stage to make the orbit circular.

   (b) Calculate the altitude h of the final orbit. Ignore air resistance and the motion of the Earth. The mass and radius of the Earth are $M_\oplus = 5.98 \times 10^{24}$ kg and $R_\oplus = 6.4 \times 10^3$ km, respectively. Let $v_0 = 6$ km/s and $\theta_0 = 30°$.

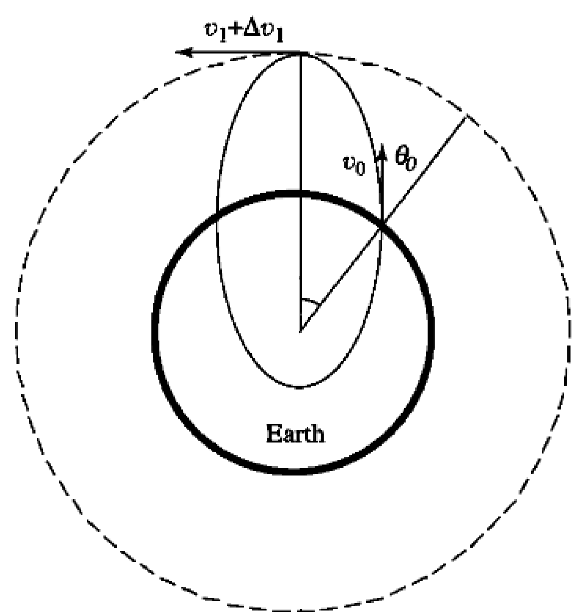

Figure P6.21 Two-stage launch to place satellite in a circular orbit.

Answer:

Here we are asked to ignore the motion of the Earth, so we assume that v_0 is the initial velocity of the rocket at the end of the (short) initial fuel burn, in an inertial frame. (In fact the rotation is an important factor adding to the initial velocity of the rocket, which is why rockets are generally launched in a westerly direction, from low-latitude locations.)

(a) Since we know both the initial velocity and the initial position, we can find the total energy and the angular momentum. Assuming that at the end of the fuel burn the rocket is still quite close to the surface of the Earth, the energy per unit mass will be

$$e \equiv \frac{E}{m} = \frac{1}{2}v_0^2 - \frac{GM_\oplus}{R_\oplus} = -\frac{1}{2}(2v_c^2 - v_0^2),$$

where $v_c = \sqrt{GM_\oplus/R_\oplus} = 7.9$ km/s is the speed of a circular orbit just above the Earth's surface. The orbital angular momentum per unit mass is

$$l \equiv \frac{L}{m} = |\boldsymbol{r} \times \boldsymbol{v}_0| = v_0 R_\oplus \sin\theta.$$

At apogee, the distance of the rocket from the centre of the Earth will be

$$r_a = a(1+\epsilon)$$

To find it we need to determine a and ϵ. These are related to the energy and angular momentum by the relations

$$e = -\frac{GM_\oplus}{2a},$$
$$l^2 = GM_\oplus a(1-\epsilon^2).$$

Therefore,

$$a = -\frac{GM_\oplus}{2e} = \frac{R_\oplus}{2-(v_0/v_c)^2},$$

For $v_0 = 6$ km/s this tells us that $a = 0.892R_\oplus$.

The eccentricity can now be found from the angular momentum equation,

$$\epsilon = \sqrt{1 - \frac{l^2}{GM_\oplus a}} = \sqrt{1 - \frac{v_0^2 R_\oplus^2 \sin^2\theta_0^2}{GM_\oplus a}} = \sqrt{1 - \left(\frac{v_0}{v_c}\right)^2 \frac{R_\oplus}{a}\sin^2\theta_0}.$$

Taking $\theta_0 = 30°$, this gives $\epsilon = 0.768$. The apogee distance is therefore $r_a = a(1+\epsilon) = 1.330R_\oplus$.

When the rocket reaches apogee, its velocity is perpendicular to $\boldsymbol{r}$, so $l = r_a v_a$. But l is conserved, so

$$v_2 = \frac{l}{r_a} = \frac{v_0 R_\oplus \sin\theta_0}{r_a} = 2.256 \text{ km/s}.$$

The speed of a circular orbit at this distance is

$$v = \sqrt{\frac{GM_\oplus}{r_a}} = \sqrt{\frac{R_\oplus}{r_a}}v_c = 6.85 \text{ km/s}.$$

So, the velocity must be increased by an amount $\Delta v_2 = v - v_2 = 4.59$ km/s.

(b) The height of the orbit (above the Earth's surface) is $r_a - R_\oplus = 0.330R_\oplus = 210$ km.