PHYS 216 Assignment 6 **Due: 15 February 2017**

1. At the Earth's north pole, a mass m is suspended by a string of length l attached to an overhead hook, in such a way that the mass can move freely in two dimensions. A researcher holds the weight so that the string makes a small angle θ with the vertical and then gives the weight a push towards the east, with sufficient speed that it swings around in a circular path, with the string keeping a constant angle with the vertical. The period T_E of this circular motion is carefully measured by the researcher. Then, the experiment is repeated except this time the initial push is to the west, giving a period T_W.

   (a) Is the period the same for both initial directions? If not, which direction will have the greater period?

   (b) If you answered "no" for part (a), calculate the relative difference in the periods:

$$\delta = \frac{T_E - T_W}{T_E + T_W}$$

   Answer:

   (a) Seen from an inertial frame, the period will be the same for either direction. However the researcher is moving with the Earth, and is therefore rotating about the North Pole, in an easterly direction. The period seen by the researcher will therefore be longer when the motion of the mass is to the east as it must catch up to the researcher. The period will be shorter when it is moving to the west as the research will be moving towards it.

   (b) To be quantitative, look at this from the inertial frame. The period is $T = 2\pi/\omega$ with $\omega = \sqrt{g/l}$. As seen by the researcher, if the mass is orbiting in an easterly direction, the angular rate will be $\omega - \omega_\oplus$ and for westerly motion the rate seen in the rotating frame will be $\omega + \omega_\oplus$. Therefore,

$$\delta = \frac{2\pi/(\omega - \omega_\oplus) - 2\pi/(\omega + \omega_\oplus)}{2\pi/(\omega - \omega_\oplus) + 2\pi/(\omega + \omega_\oplus)} = \frac{\omega_\oplus}{\omega}.$$

   The same result can also be obtained using the formalism developed Section 5. Choose a frame O', rotating with the Earth, so that the mass traces a circular path of radius r' centred on the origin. The period seen in an inertial frame is $T = 2\pi r/v = 2\pi r'/v$. The period seen in the rotating frame is $T' = 2\pi/v'$. The two are related by

$$\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{\omega}_\oplus \times \boldsymbol{r}'.$$

   For easterly motion, $\boldsymbol{v}'$ is in the same direction as $\boldsymbol{\omega}_\oplus \times \boldsymbol{r}'$, but for westerly motion it is opposite. Therefore,

$$\frac{2\pi r'}{T} = \frac{2\pi r'}{T_E} + \omega_\oplus r' = \frac{2\pi r'}{T_W} - \omega_\oplus r'$$

   From this it follows that $T_E + T_W = 2T_E T_W/T = \omega T_E T_W/\pi$ and $T_E - T_W = \omega_\oplus T_E T_W/\pi$, so $\delta = \omega_\oplus/\omega$ as before.

2. An artillery shell is fired vertically from the equator with an initial velocity of 1000 m/s. Because of the Coriolis force, it drifts horizontally. Ignoring air resistance, find the amount and direction of the drift when the shell reaches its highest point. Find the amount and

direction of the drift when it hits the ground. What is the maximum height reached by the shell?

Answer:

The position of the shell is given by the general result for ballistic motion

$$\boldsymbol{r}' = \boldsymbol{r}'_0 + \boldsymbol{v}'_0 t + \frac{1}{2}\boldsymbol{g}' t^2 - \boldsymbol{\omega} \times \boldsymbol{v}'_0 t^2 - \frac{1}{3}\boldsymbol{\omega} \times \boldsymbol{g}' t^3.$$

now $\boldsymbol{v}_0 = v_0\boldsymbol{k}$, $\boldsymbol{g}' = v'_0\boldsymbol{k}$ and we can choose $\boldsymbol{r}'_0 = 0$. Therefore, in a coordinate system in which z' is vertical, x' is east and y' north, the components of this equation are

$$\begin{aligned} x' &= -\omega_{\oplus} v'_0 \cos\lambda t^2 + \frac{1}{3}\omega_{\oplus} g' \cos\lambda t^3. \\ y' &= 0 \\ z' &= v'_0 t - \frac{1}{2} g' t^2. \end{aligned}$$

By setting $dz'/dt = 0$ we see that the shell reaches its highest point when $t = v'_0/g$. The drift at this time is

$$x'_{\text{top}} = -\frac{\omega_{\oplus} {v'_0}^3 \cos\lambda}{g'^2}\left(1 - \frac{1}{3}\right) = -\frac{2\omega_{\oplus} {v'_0}^3 \cos\lambda}{3g'^2}.$$

which is a drift to the west.

The time when it reaches the ground is found by setting $z' = 0$, which gives $t = 2v'_0/g$ (showing that the shell takes as long to come down as it takes to go up). The drift at this time is

$$x'_{\text{ground}} = -\frac{\omega_{\oplus} {v'_0}^3 \cos\lambda}{g'^2}\left(4 - \frac{8}{3}\right) = -\frac{4\omega_{\oplus} {v'_0}^3 \cos\lambda}{3g'^2}.$$

which is twice the drift at the highest point.

Putting in numbers for a location at the equator ($\cos\lambda = 1$) we find

$$\begin{aligned} x'_{\text{top}} &= -\frac{2(7.29 \times 10^{-5})(1000)^3}{3(9.81)^2} = -505 \text{ m}, \\ x'_{\text{ground}} &= -1010 \text{ m}. \end{aligned}$$

The shell reaches a maximum height of ${v'_0}^2/2g' = 51$ km.

3. The force on a charged particle in an electric field $\boldsymbol{E}$ and a magnetic field $\boldsymbol{B}$ is given by the *Lorentz force*

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

where q is the particle's electric charge and v is its velocity in an inertial system. Show that the equation of motion in a frame rotating with angular velocity $\boldsymbol{\omega} = -q\boldsymbol{B}/2m$ is, for small $\boldsymbol{B}$.

$$m\boldsymbol{a}' = q\boldsymbol{E}.$$

In this frame, the term involving $\boldsymbol{B}$ is eliminated. This result is known as *Larmor's theorem*.

Answer:

Let the rotating frame O' and the inertial frame O have the same origin. The equation of motion in the rotating frame is then

$$m\boldsymbol{a}' = \boldsymbol{F}' = \boldsymbol{F} - 2m\boldsymbol{\omega} \times \boldsymbol{v}' - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}').$$

(The terms involving $\boldsymbol{A}_0$ and $\dot{\boldsymbol{\omega}}$ do not appear because these quantities are zero.) Also, recall that

$$\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{\omega} \times \boldsymbol{r}'.$$

so

$$\boldsymbol{F} = q\left[\boldsymbol{E} + (\boldsymbol{v}' + \boldsymbol{\omega} \times \boldsymbol{r}') \times \boldsymbol{B}\right].$$

If we now $\boldsymbol{\omega} = -q\boldsymbol{B}/2m$ this becomes

$$\boldsymbol{F} = q\left[\boldsymbol{E} + \boldsymbol{v}' \times \boldsymbol{B} - \frac{q}{2m}(\boldsymbol{B} \times \boldsymbol{r}') \times \boldsymbol{B}\right] = q\left[\boldsymbol{E} + \boldsymbol{v}' \times \boldsymbol{B} + \frac{q}{2m}\boldsymbol{B} \times (\boldsymbol{B} \times \boldsymbol{r}')\right].$$

Substituting this into the equation of motion, we get

$$\begin{aligned} m\boldsymbol{a}' &= q\left[\boldsymbol{E} + \boldsymbol{v}' \times \boldsymbol{B} + \frac{q}{2m}\boldsymbol{B} \times (\boldsymbol{B} \times \boldsymbol{r}')\right] + q\boldsymbol{B} \times \boldsymbol{v}' - \frac{q^2}{4m}\boldsymbol{B} \times (\boldsymbol{B} \times \boldsymbol{r}'), \\ &= q\boldsymbol{E} + \frac{q^2}{4m}\boldsymbol{B} \times (\boldsymbol{B} \times \boldsymbol{r}'). \end{aligned}$$

The second term is quadratic in B, so if B is sufficiently small (specifically, if $qB^2 \ll mE/r'$) it can be ignored.