PHYS 216 Assignment 5 **Due: 8 February 2017**

1. By finding the curl, determine which of the following forces are conservative:

   (a) $\boldsymbol{F} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$

   (b) $\boldsymbol{F} = y\boldsymbol{i} - x\boldsymbol{j} + z^2\boldsymbol{k}$

   (c) $\boldsymbol{F} = y\boldsymbol{i} + x\boldsymbol{j} + z^3\boldsymbol{k}$

   (d) $\boldsymbol{F} = -kr^{-n}\boldsymbol{e}_r$ in spherical coordinates

   Answer:

   (a), (c), (d)

2. Particles of mud are thrown from the rim of a rolling bicycle wheel. If the forward speed of the bicycle is v_0, and the radius of the wheel is b, show that the greatest height above the ground that the mud can go is

$$b + \frac{v_0^2}{2g} + \frac{gb^2}{2v_0^2}.$$

   At what point on the rolling wheel does this mud leave? (Note: it is necessary to assume that $v_0^2 \geq bg$.)

   Answer:

   The problem is two dimensional. Let x be the horizontal position and z the height above ground. Let θ be the angle between a point on the rim of the wheel and the point of contact at the bottom of the wheel, as seen from the centre of the wheel. This angle increases as the wheel rotates. The position vector of the centre of the wheel is $(v_0 t, b)$. To this we add the vector $(-b\sin\theta, b\cos\theta)$ that extends from the centre of the wheel to the point on the rim.

$$x = v_0 t - b\sin\theta,$$
$$z = b - b\cos\theta,$$

   The wheel makes one complete rotation when it travels a distance $2\pi b$, which takes a time $2\pi b/v_0$. So the relation between angle and time is

$$\theta = \omega t = v_0 t/b.$$

   We can now find the velocity components by differentiation,

$$\dot{x} = v_0(1 - \cos\omega t),$$
$$\dot{z} = v_0 \sin\omega t,$$

   Mud leaving the wheel at angle θ will have a vertical velocity component $v_0\sin\theta$ and an initial height of $b(1-\cos\theta)$. From the example worked out in class we know that a projectile fired vertically with initial speed v reaches a maximum height $v^2/2g$ above its initial position. So the the height that the mud can reach is

$$z = b(1-\cos\theta) + \frac{v_0^2\sin^2\theta}{2g} = b(1-\cos\theta) + \frac{v_0^2(1-\cos^2\theta)}{2g}$$

To find the value of θ that maximizes this we set $dz/d\cos\theta = 0$,

$$-b - \frac{v_0^2 \cos\theta}{g} = 0$$

so

$$\cos\theta = -\frac{bg}{v_0^2}.$$

Substituting this back in the equation for z gives the maximum possible height

$$z_{\rm max} = b\left(1 + \frac{bg}{v_0^2}\right) + \frac{v_0^2(1 - b^2g^2/v_0^4)}{2g} = b + \frac{b^2 g}{v_0^2} + \frac{v^2}{2g} - \frac{b^2 g}{2v_0^2} = b + \frac{v^2}{2g} + \frac{b^2 g}{2v_0^2}.$$

The angle θ that this mud would need to leave the wheel is

$$\theta = \arccos\left(-\frac{bg}{v_0^2}\right) = \pi - \arccos\left(\frac{bg}{v_0^2}\right).$$

3. A cannon that is capable of firing a shell at speed v_0 is mounted on a vertical tower of height h that overlooks a level plain below. Ignoring air resistance, show that the elevation angle α at which the cannon must be set to achieve maximum range is given by the expression

$$\csc^2\alpha = 2\left(1 + \frac{gh}{v_0^2}\right).$$

Answer:

This is similar to the example worked out in class. The equations of motion are

$$\ddot{x} = 0, \quad \ddot{z} = -g,$$

and the initial conditions at time $t = 0$ are

$$x_0 = 0, \quad z_0 = h, \quad \dot{x}_0 = v_0\cos\alpha, \quad \dot{z}_0 = v_0 \sin\alpha.$$

The velocity and position can be found by direct integration,

$$\begin{aligned} \dot{x} &= \dot{x}_0, \\ \dot{z} &= -gt + \dot{z}_0, \\ x &= \dot{x}_0 t, \\ z &= -\frac{1}{2}gt^2 + \dot{z}_0 t + h, \end{aligned}$$

The time at which the projectile hits the ground is found by setting $z = 0$ in the last equation, and solving for t,

$$gt^2 - 2\dot{z}_0 t - 2h = 0,$$

which has the solution

$$t = \frac{1}{g}\left(\dot{z}_0 \pm \sqrt{\dot{z}_0^2 + 2gh}\right)$$

We must take the positive square root to get $t > 0$. The range is now found by substituting this value for t in the equation for x,

$$x = \frac{\dot{x}_0}{g}\left(\dot{z}_0 + \sqrt{\dot{z}_0^2 + 2gh}\right).$$

In terms of the angle α this is

$$x = \frac{v_0 \cos\alpha}{g} \left(v_0 \sin\alpha + \sqrt{v_0^2 \sin^2\alpha + 2gh} \right) = \frac{v_0}{2g} \left(v_0 \sin 2\alpha + \sqrt{v_0^2 \sin^2 2\alpha + 8gh\cos^2\alpha} \right).$$

To find the maximum range, set $dx/d\alpha = 0$ and solve for α.

$$2v_0 \cos 2\alpha + \frac{2v_0^2 \sin 2\alpha \cos 2\alpha + 4gh \sin 2\alpha}{\sqrt{v_0^2 \sin^2 2\alpha + 8gh \cos^2\alpha}} = 0$$

so

$$v_0^2 \sin 2\alpha + 2gh \tan 2\alpha = -v_0 \sqrt{v_0^2 \sin^2 2\alpha + 8gh \cos^2\alpha}.$$

Squaring both sides and simplifying, we get

$$\begin{aligned}
v_0^4 \sin^2 2\alpha + 4g^2h^2 \tan^2 2\alpha + 4v_0^2 gh \sin 2\alpha \tan 2\alpha &= v_0^4 \sin^2 2\alpha + 8v_0^2 gh \cos^2\alpha, \\
gh \tan^2 2\alpha + v_0^2 \sin 2\alpha \tan 2\alpha &= 2v_0^2 \cos^2\alpha, \\
\frac{\sin 2\alpha \tan 2\alpha - 2\cos^2\alpha}{\tan^2 2\alpha} &= -\frac{gh}{v_0^2}, \\
\cos 2\alpha - \frac{\cos^2 2\alpha}{2\sin^2\alpha} &= \frac{gh}{v_0^2}, \\
2\sin^2\alpha - 1 - \frac{(2\sin^2\alpha - 1)^2}{2\sin^2\alpha} &= -\frac{gh}{v_0^2}, \\
\frac{4\sin^2\alpha - 1}{2\sin^2\alpha} &= 1 + \frac{gh}{v_0^2}, \\
2 - \frac{\csc^2\alpha}{2} &= 1 - \frac{gh}{v_0^2},
\end{aligned}$$

Therefore,

$$\csc^2\alpha = 2\left(1 + \frac{gh}{v_0^2}\right).$$