PHYS 216 Assignment 4 **Due: 1 February 2017**

1. [10 points] Show that the ratio of two successive maxima in the displacement of a damped harmonic oscillator is constant.

   Answer:

   Since there are oscillations, the harmonic oscillator must be underdamped. We saw that the solution for the displacement can be written as

$$x = Ae^{-\gamma t}\cos(\omega_r t + \phi_0)$$

   Maximum (positive or negative) displacement will occur when the velocity is zero, so we take the derivative of x and set it equal to zero,

$$v = \ddot{x} = -Ae^{-\gamma t}(\omega_d \sin(\omega_d t + \phi_0) + \gamma \sin(\omega_d t + \phi_0)] = 0.$$

   This has the solution

$$\tan(\omega_d t + \phi_0) = \frac{\gamma}{\omega_d}.$$

   The tangent function is periodic, with period π, so if t is a solution of this equation, then so is $t+n\pi/\omega_d$ where n is any positive integer. This shows that successive maxima are all separated by the same time interval, $\Delta t = \pi/\omega_d$. Successive positive maxima will be separated by an interval of $2\Delta t$.

   To find the ratio of successive maxima we need to compute

$$\frac{x(t+\Delta t)}{x(t)} = \frac{e^{-\gamma(t+\Delta t)}\cos(\omega_r t + \phi_0 + \pi)}{e^{-\gamma t}\cos(\omega_r t + \phi_0)} = -e^{-\gamma \Delta t}, \quad (1)$$

   which is constant.

2. [20 points] A simple pendulum whose length $l = 9.8$ m satisfies the equation

$$\ddot{\theta} + \sin\theta = 0 \quad (2)$$

   (a) If θ_0 is the amplitude of the oscillation, show that its period T is given by

$$T = 4\int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \alpha \sin^2\phi}} \quad (3)$$

   where

$$\alpha = \sin^2(\theta_0/2). \quad (4)$$

   Hint: the identity $\cos\theta = 1 - 2\sin^2(\theta/2)$ will be helpful.

   (b) Expand the integrand in powers of α, integrate term by term, and fond the period T as a power series in α. Keep turns up to and including $O(\alpha^2)$.

   (c) Expand α in a power series of θ_0, insert the result into the power series found in (b), and find the period T as a power series in θ_0. Keep terms up to and including $O(\theta_0^2)$.

   Answers:

(a) The equation of motion can be expressed in terms of the angular velocity $\omega = \dot{\theta}$ as follows

$$\ddot{\theta} = \frac{d\omega}{dt} = \frac{d\theta}{dt}\frac{d\omega}{d\theta} = \omega\frac{d\omega}{d\theta} = \frac{d}{d\theta}\left(\frac{1}{2}\omega^2\right) = -\sin\theta.$$

Integrating this we find

$$\omega^2 = -2\int \sin\theta d\theta = 2\cos\theta + const$$

Since $\omega = 0$ at the turning point where $\theta = \theta_0$, the constant must be equal to $-2\cos\theta_0$, so

$$\omega^2 = 2(\cos\theta - \cos\theta_0)$$

The period is four times the time needed to go from $\theta = 0$ to $\theta = \theta_0$.

$$T = \int dt = \int \frac{d\theta}{\omega} = 4\int_0^{\Theta_0} \frac{d\theta}{\omega} = 4\int_0^{\theta_0} \frac{d\theta}{\sqrt{2(\cos\theta - \cos\Theta_0)}}.$$

Using the trig identity $\cos\theta = 1 - 2\sin^2(\theta/2)$, this can be written as

$$\begin{aligned} T &= 2\int_0^{\theta_0} \frac{d\theta}{\sqrt{\sin^2(\theta_0/2) - \sin^2(\theta/2)}}, \\ &= \frac{2}{\sin(\theta_0/2)}\int_0^{\Theta_0} \frac{d\theta}{\sqrt{1 - \sin^2(\theta/2)/\sin^2(\theta_0/2)}}. \end{aligned}$$

To simplify the denominator we try the substitution

$$\sin\phi = \frac{\sin(\theta/2)}{\sin(\theta_0/2)}.$$

so

$$\cos\phi d\phi = \frac{\cos(\theta/2)d\theta}{2\sin(\theta_0/2)}.$$

When $\theta = 0$, $\phi = 0$ and when $\theta = \theta_0$, $\sin\phi = 1$, so the limits of integration change to $(0, \phi/2)$. Putting this in and simplifying, we get

$$T = 4\int_0^{\pi/2} \frac{d\phi}{\cos(\theta/2)} = 4\int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \alpha\sin^2\phi}}.$$

(b) Using the expansion

$$(1+x)^\beta = 1 + \beta x + \frac{\beta(\beta-1)}{2!}x^2 + \frac{\beta(\beta-1)(\beta-2)}{3!}x^3 + \cdots,$$

the integral becomes

$$\begin{aligned} T &= 4\int_0^{\pi/2} \left(1 - \alpha\sin^2\phi\right)^{-1/2} d\phi, \\ &= 4\int_0^{\pi/2} \left(1 + \frac{1}{2}\alpha\sin^2\phi + \cdots\right) d\phi. \end{aligned}$$

The integral can be done using the identity

$$\sin^2\phi = \frac{1}{2}(1-\cos 2\phi).$$

We get

$$\begin{aligned} T &= 4\int_0^{\pi/2}\left[1+\frac{\alpha}{4}(1-\cos 2\phi)+\cdots\right]d\phi, \\ &= 4\left[\left(1+\frac{\alpha^2}{4}\right)\phi - \frac{\alpha}{8}\sin 2\phi\right]_0^{\pi/2} = 2\pi\left(1+\frac{\alpha}{4}\right). \end{aligned}$$

(c) Using

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots,$$

we obtain

$$\begin{aligned} \alpha = \sin^2(\theta_0/2) &= \left[\frac{\theta_0}{2} - \frac{\theta_0^3}{48} + \cdots\right]^2, \\ &= \frac{\theta_0^2}{4}\left[1-\frac{\theta_0^2}{24}+\cdots\right]^2 = \frac{\theta_0^2}{4}\left[1-\frac{\theta_0^2}{12}\right]+\cdots \end{aligned}$$

So, to second order in θ_0, the period is given by

$$T = 2\pi\left(1+\frac{\theta_0^2}{16}+\cdots\right),$$

which shows that the period increases with increasing amplitude.