PHYS 216 Assignment 2 **Due: 18 January 2017**

1. Prove that
$$\frac{d}{dt}[\boldsymbol{r} \cdot (\boldsymbol{v} \times \boldsymbol{a})] = \boldsymbol{r} \cdot (\boldsymbol{v} \times \dot{\boldsymbol{a}}). \tag{1}$$

   Answer:

   Using the chain rule,
$$\begin{aligned}\frac{d}{dt}[\boldsymbol{r} \cdot (\boldsymbol{v} \times \boldsymbol{a})] &= \dot{\boldsymbol{r}} \cdot (\boldsymbol{v} \times \boldsymbol{a}) + \boldsymbol{r} \cdot (\dot{\boldsymbol{v}} \times \boldsymbol{a}) + \boldsymbol{r} \cdot (\boldsymbol{v} \times \dot{\boldsymbol{a}}), \\ &= \boldsymbol{v} \cdot (\boldsymbol{v} \times \boldsymbol{a}) + \boldsymbol{r} \cdot (\boldsymbol{a} \times \boldsymbol{a}) + \boldsymbol{r} \cdot (\boldsymbol{v} \times \dot{\boldsymbol{a}}),\end{aligned}$$

   The first term on the RHS is zero because $\boldsymbol{v} \times \boldsymbol{a}$ is perpendicular to $\boldsymbol{v}$. The second term vanishes because $\boldsymbol{a} \times \boldsymbol{a} = 0$. This gives the desired result.

2. Suppose that the force acting on a particle of mass m is given by
$$F = kvx \tag{2}$$
   in which k is a positive constant. If the particle passes through the origin ($x = 0$) at time $t = 0$ with velocity v_0, find x as a function of t.

   Answer:

   The equation of motion is
$$m\ddot{x} = m\frac{dv}{dt} = mv\frac{dv}{dx} = kvx. \tag{3}$$
   This simplifies to
$$m\frac{dv}{dx} = kx. \tag{4}$$
   which can be separated to give
$$dv = \frac{k}{m}xdx. \tag{5}$$
   Integrating both sides we have
$$v = \frac{k}{2m}x^2 + v_0, \tag{6}$$
   where the integration constant v_0 is clearly the speed at $x = 0$, as required.

   Since $v = dx/dt$, we can again separate varibles
$$\frac{dx}{kx^2/2m + v_0} = dt. \tag{7}$$

   This can be integrated to give
$$t = \frac{1}{v_0}\int_0^x \frac{dx}{kx^2/2mv_0 + 1} = \sqrt{\frac{2m}{kv_0}}\int_0^x \frac{du}{1+u^2} = \sqrt{\frac{2m}{kv_0}}\arctan\left(\sqrt{\frac{k}{2mv_0}}x\right) \tag{8}$$

   To solve this for $x(t)$, multiply by $\sqrt{kv_0/2m}$ and take the tangent of both sides,
$$x = \sqrt{\frac{2mv_0}{k}}\tan\left(\sqrt{\frac{kv_0}{2m}}t\right). \tag{9}$$

   The particle goes to infinity in a finite time $t = (\pi/2)\sqrt{2m/kv_0}$.

3. A particle of mass m is released from rest a distance b from a fixed origin of force that attracts the particle according to the inverse square law:

$$F(x) = -kx^{-2}. \tag{10}$$

Show that the time taken for the particle to reach the origin is

$$t = \pi\sqrt{\frac{mb^3}{8k}} \tag{11}$$

Answer:

It is easy to verify that the potential,

$$V(x) = -\frac{k}{x} \tag{12}$$

gives rise to the desired force, since

$$F(x) = -\frac{dV}{dx} = -\frac{k}{x^2} \tag{13}$$

Initially, when $x = b$, the kinetic energy T is zero, so the total energy is

$$E = T + V = -\frac{k}{b} \tag{14}$$

The velocity is then

$$v(x) = -\sqrt{\frac{2}{m}(E - V)} = -\sqrt{\frac{2k}{m}\left(\frac{1}{x} - \frac{1}{b}\right)} = -\sqrt{\frac{2k(b - x)}{mbx}} \tag{15}$$

(We need the negative square root since the particle is moving in the $-x$ direction.) We can now find the time by integrating the reciprocal of the velocity,

$$t = \int dt = \int_b^0 \frac{dx}{v(x)} = -\sqrt{\frac{mb}{2k}} \int_b^0 \sqrt{\frac{x}{b - x}}\, dx = \sqrt{\frac{mb}{2k}} \int_0^b \sqrt{\frac{x}{b - x}}\, dx. \tag{16}$$

To do the integral, try the substitution $x = b\sin^2\theta$. Then $dx = 2b\sin\theta\cos\theta d\theta$. The limits of integration change to 0 to $\pi/2$. The integral becomes

$$\int_0^b \sqrt{\frac{x}{b - x}}\, dx = \int_0^{\pi/2} \frac{2b\sin^2\theta\cos\theta d\theta}{\sqrt{1 - \sin\theta^2}} = 2b\int_0^{\pi/2} \sin^2\theta d\theta. \tag{17}$$

Now simplify this by using the trigonometric identity $\sin^2\theta = (1 - \cos 2\theta)/2$. we see that

$$\int_0^{\pi/2} (1 - \cos 2\theta) d\theta = \frac{\pi}{2} - \int_0^{\pi/2} \cos 2\theta\, d\theta = \frac{\pi}{2}. \tag{18}$$

which gives the desired result

$$t = \pi\sqrt{\frac{mb^3}{8k}}. \tag{19}$$