**PHYS 216 Assignment 1** **Due: 11 January 2017**

1. Evaluate the dimensions of both sides of the following three equations, reducing them to the simplest ratios of the basic dimensions of $[L]$, $[M]$ and $[T]$, and verify that the dimensions match. For example,

$$[pressure] = \left[\frac{force}{area}\right] = \left[\frac{ML}{T^2L^2}\right] = \frac{[M]}{[L][T]^2}.$$

$$c_s = \sqrt{P/\rho},$$
$$t_{ff} = \sqrt{\frac{3\pi}{32G\rho}},$$
$$\omega = \sqrt{k/m}$$

Here $c_s$ is the speed of sound in a gas, $P$ is the gas pressure, $\rho$ is the mass density, $t_{ff}$ is the gravitational free-fall time, $G$ is Newton's constant, $\omega$ is the oscillation frequency of a mass on a spring, $m$ is mass, and $k$ is the spring constant (Newtons per metre).

Answer:

$$\begin{aligned}
[c_s] &= \left[\frac{L}{T}\right] = \frac{[L]}{[T]}, \\
[P] &= \left[\frac{ML}{T^2L^2}\right] = \frac{[M]}{[L][T]^2}, \\
[\rho] &= \left[\frac{M}{L^3}\right] = \frac{[M]}{[L]^3}, \\
[G] &= \left[\frac{L^2}{T^2M}\right] = \frac{[L]^3}{[T]^2[M]}, \\
[k] &= \left[\frac{ML}{T^2L}\right] = \frac{[M]}{[T]^2}
\end{aligned} \tag{1}$$

Therefore the dimensions of three equations given are

$$\frac{[L]}{[T]} = \sqrt{\frac{[M][L]^3}{[L][T]^2[M]}} = \frac{[L]}{[T]},$$
$$[T] = \sqrt{\frac{[T]^2[M][L]^3}{[L]^3[M]}} = [T],$$
$$\frac{1}{[T]} = \sqrt{\frac{[M]}{[T]^2[M]}} = \frac{1}{[T]}.$$

2. Using the definitions of the dot and cross products in terms of Cartesian components, prove that

$$\boldsymbol{A}\cdot\boldsymbol{B}\times\boldsymbol{C} = \boldsymbol{B}\cdot\boldsymbol{C}\times\boldsymbol{A} = \boldsymbol{C}\cdot\boldsymbol{A}\times\boldsymbol{B}$$

Answer:

The first expression is

$$\begin{aligned}A_x(B_yC_z - B_zC_y) + A_y(B_zC_x - B_xC_z) + A_z(B_xC_y - B_yC_x) = \\ A_xB_yC_z + B_xC_yA_z + C_xA_yB_z - A_xC_yB_z - B_xA_yC_z - C_xB_yA_z\end{aligned}$$

The second expression is obtained from the first by a cyclic permutation $A \to B$, $B \to C$, $C \to A$. Observe that the first three terms in the last line of the expansion are just cyclic permutations of each other. So the sum of the first three terms will be unchanged by such a permutation. Similarly, the last three terms are also cyclic permutations of each other so their sum will also be unchanged. Therefore the second expression gives the same expansion. Similarly the third expression is just a cyclic permutation of the second expression and will also have the same expansion. So all three expressions must be equal.

3. What is the value of $x$ that makes the following transformation orthogonal?

$$\boldsymbol{R} = \begin{pmatrix} x & 0 & x \\ 0 & 1 & 0 \\ -x & 0 & x \end{pmatrix} \tag{2}$$

What transformation is represented by $\boldsymbol{R}$?

Answer:

The transpose of $\boldsymbol{R}$ is

$$\boldsymbol{R}^T = \begin{pmatrix} x & 0 & -x \\ 0 & 1 & 0 \\ x & 0 & x \end{pmatrix} \tag{3}$$

Orthogonal means that $\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I}$. Working out the first matrix product we find

$$\boldsymbol{R}^T\boldsymbol{R} = \begin{pmatrix} x & 0 & -x \\ 0 & 1 & 0 \\ x & 0 & x \end{pmatrix} \begin{pmatrix} x & 0 & x \\ 0 & 1 & 0 \\ -x & 0 & x \end{pmatrix} = \begin{pmatrix} 2x^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2x^2 \end{pmatrix} \tag{4}$$

In order for this to equal the unit matrix, we must have $2x^2 = 1$, so $x = \pm 1/\sqrt{2}$.

Now $\sin(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$ and $\sin(5\pi/4) = \cos(5\pi/4) = -1/\sqrt{2}$, so the two solutions correspond to the rotation matrices

$$\begin{aligned}\boldsymbol{R}_+ &= \begin{pmatrix} \cos(\pi/4) & 0 & \sin(\pi/4) \\ 0 & 1 & 0 \\ -\sin(\pi/4) & 0 & \cos(\pi/4) \end{pmatrix}, \\ \boldsymbol{R}_- &= \begin{pmatrix} \cos(5\pi/4) & 0 & \sin(5\pi/4) \\ 0 & 1 & 0 \\ -\sin(5\pi/4) & 0 & \cos(5\pi/4) \end{pmatrix},\end{aligned} \tag{5}$$

which represent rotations about the $y$ axis by an angles of 45° and 225° respectively.