**PHYS 216 Assignment 10** **Due: 29 March 2017**

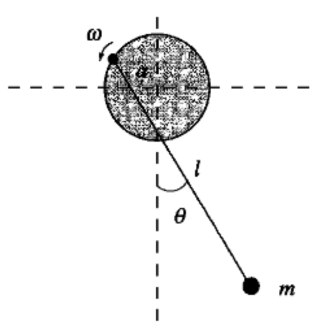
1. Use Lagrange's method to find the equations of motion for a double harmonic oscillator, consisting of a spring of stiffness $k_1$ suspending a mass $m_1$, from which hangs another spring of stiffness $k_2$ supporting a mass $m_2$.

   Answers:

   To specify the configuration of the system at any time, we need two generalized coordinates, which we can take to be the positions of the two masses, $x_1$ and $x_2$. Therefore there are two degrees of freedom and there will be two Lagrange equations.

   The kinetic energy is
$$T = \frac{1}{2}m_1\dot{x_1}^2 + \frac{1}{2}m_1\dot{x_2}^2.$$
   The potential energy is
$$V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_2 - x_1)^2,$$
   so the Lagrangian is
$$L = \frac{1}{2}(m_1\dot{x_1}^2 + m_1\dot{x_2}^2) - \frac{1}{2}\left[k_1x_1^2 + k_2(x_2 - x_1)^2\right]$$
   The Lagrange equations are
$$\begin{aligned} m_1\ddot{x_1} + k_1x_1 - k_2(x_2 - x_1) &= 0, \\ m_2\ddot{x_2} + k_2(x_2 - x_1) &= 0. \end{aligned}$$
   These are identical to the Newtonian equations of motion.

2. A simple pendulum of length $l$ and mass $m$ is suspended from a point on the circumference of a massless disk of radius $a$ that rotates with an constant angular velocity $\omega$ about its axis, as shown below. Use Lagrangian mechanics to find the equation of motion of the mass.




   Answer:

   Let $\phi = \omega t$ be the the rotation angle of the disk, measured counterclockwise from the highest point, and $\theta$ is the angle of the pendulum with respect to the vertical direction. Since $\phi$ is a known function of time, there is only one degree of freedom, represented by $\theta$.

   To find the kinetic energy we need to find the velocity of the mass (in an inertial frame). In a Cartesian coordinate system with origin at the axle of the disk, the position of the pendulum mass is
$$x = -a\sin\phi + l\sin\theta, \quad y = a\cos\phi - l\cos\theta. \tag{1}$$

The velocity components are

$$\dot{x} = -a\dot{\phi}\cos\phi + l\dot{\theta}\cos\theta, \quad \dot{y} = -a\dot{\phi}\sin\phi + l\dot{\theta}\sin\theta. \tag{2}$$

The kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m[a^2\dot{\phi}^2 + l^2\dot{\theta}^2 - 2al\dot{\phi}\dot{\theta}(\cos\phi\cos\theta + \sin\phi\sin\theta)] \\ &= \frac{1}{2}m[a^2\dot{\phi}^2 + l^2\dot{\theta}^2 - 2al\dot{\phi}\dot{\theta}\cos(\theta - \phi)] \end{aligned} \tag{3}$$

The potential is

$$V = mgy = mg(a\cos\phi - l\cos\theta).$$

So the Lagrangian is

$$L = \frac{1}{2}m[a^2\omega^2 + l^2\dot{\theta}^2 - 2al\omega\dot{\theta}\cos(\theta - \omega t)] - mg(a\cos\omega t - l\cos\theta).$$

The Lagrange equation is,

$$ml^2\ddot{\theta} - mal\omega^2\sin(\theta - \omega t) + mgl\sin\theta = 0,$$

which simplifies to

$$l\ddot{\theta} = -g\sin\theta + a\omega^2\sin(\theta - \omega t),$$

This is an example of a system in which the Lagrangian is an explicit function of time, in addition to the generalized coordinates and velocities.

3. The equation of motion of a particle of rest mass $m$ moving a relativistic speed in a potential $V$ is

$$m\gamma\frac{d}{dt}(\gamma\boldsymbol{v}) = \boldsymbol{f},$$

where $\boldsymbol{f} = -\gamma\boldsymbol{\nabla}V$ is the relativistic force, $\gamma = 1/\sqrt{1 - v^2/c^2}$ is the *Lorentz factor* and $c$ is the speed of light. Show that the Lagrangian

$$L = -mc^2/\gamma - V$$

gives this equation of motion.

Answer:

The particle is moving in three dimensions, with no constraints. Therefore, there are three generalized coordinates, which we take to be $x, y, z$.

In terms of these, the Lagrangian is

$$L = -mc^2\left[1 - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)/c^2\right]^{1/2} - V.$$

Thus,

$$\begin{aligned} \frac{\partial L}{\partial \dot{x}} &= m\left[1 - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)/c^2\right]^{-1/2}\dot{x}, \\ \frac{\partial L}{\partial x} &= -\frac{\partial V}{\partial x}. \end{aligned} \tag{4}$$

The Lagrange equation corresponding to $x$ is therefore

$$m\frac{d}{dt}(\gamma\dot{x}) = -\frac{\partial V}{\partial x}.$$

There are corresponding equations for $y$ and $z$. We recognize these as the components of the vector equation

$$m\frac{d}{dt}(\gamma\boldsymbol{v}) = -\boldsymbol{\nabla}V.$$

Multiplying this by $\gamma$ and substituting $\boldsymbol{f} = -\gamma\boldsymbol{\nabla}V$ gives the desired equation.