PHYS 216

Intermediate Mechanics

Lecture Notes

Paul Hickson

2017-01-01

Contents

1 Introduction

The aim of this course is to present the essential concepts of classical mechanics at an intermediate level. It is assumed that the student has already learned basic concepts such as reference frames, vectors, forces and Newtons laws of motion. We build on this to study motion in three dimensions, solid body dynamics, damped and driven oscillators, celestial mechanics and end with an introduction to Lagrangian and Hamiltonian mechanics.[1]

We begin with a review of some basic concepts.

1.1 Units

One distinguishes between pure numbers such as 1, 2, π, e, etc. which we are ***dimensionless***, and ***dimensional*** quantities that require physical units for their specification. For example a measurement of distance is meaningless until the units are specified (2.5 cm is not the same distance as 2.5 m).

Of fundamental importance are the measurements of distance and time. To these we must add a unit describing mass in order to discuss the acceleration of bodies to which forces are applied. In the Systèm Internationale (SI) system of units, the base units for these quantities are the ***metre***, ***second*** and ***kilogram***. These are the basic units of mechanics.

When electric or magnetic forces are present, one needs a new base unit, which is taken to be the ***ampere***, describing electrical current. When considering processes involving heat, a unit of temperature, the ***kelvin***, is added.

Other systems of units are often used, such as the *cgs* system. We will use SI units exclusively and convert other units to SI as necessary.

1.2 Dimensional analysis

The word *dimension* can have several meanings. In this case it refers to the units of a quantity. For example, in the SI system, length does not have the same units as time.

Square brackets are used to indicate the dimension of a quantity. For example, the dimensions of length, time and mass are written $[L]$, $[T]$, $[M]$. The dimensions of other quantities can be reduced to combinations of the basic units. For acceleration, we have

$$[a] = \left[\frac{L}{T^2}\right] = [L][T]^{-2}. \tag{1.1}$$

When deriving equations, it is worth checking to be sure that the dimensions of both sides of the equation are the same. If they are not, the equation is incorrect. Of course equality of units does not guarantee that the equation is correct. It is a necessary but not a sufficient condition.

[1] Many figures in these notes are reproduced from the text book *Analytical Mechanics* by Fowles and Cassiday and are copyright. They appear with their original figure numbers and captions.

Dimensional analysis is a simple but powerful technique that can give insight into physical laws. In many cases, simple relationships between physical parameters can be guessed, to within a constant of proportionality, just by considering the units.

As an example, consider waves on the ocean. These come in various sizes which we can characterize by the wavelength λ. You might have noticed that the speed with which a wave travels depends on its wavelength. Waves that have a large wavelength propagate faster. This is unlike the situation for sound, and light waves, for which the velocity is practically independent of wavelength.

Lets try to guess the relationship between the propagation speed v and the wavelength λ of a wave. We begin by considering what quantities might be relevant. Since the waves oscillate due to the restoring force provided by gravity, we expect that the gravitational acceleration g at the Earth's surface, will play a role.

Lets write our proposed formula in the form

$$v = \alpha g^{\beta} \lambda^{\gamma} \tag{1.2}$$

where α, β, γ are dimensionless numbers. The units of g are $[L][T]^{-2}$ and the units of v are $[L][T]^{-1}$. So, if we take the dimensions of both sides of the equation we find

$$[L][T]^{-1} = [L]^{\beta}[T]^{-2\beta}[L]^{\gamma} = [L]^{\beta+\gamma}[T]^{-2\beta} \tag{1.3}$$

In order to match the powers of $[T]$ on both sides, we need $\beta = 1/2$. Then, the powers of $[L]$ require that $\gamma = 1/2$. Our formula becomes

$$v = \alpha\sqrt{g\lambda} \tag{1.4}$$

Dimensional analysis cannot tell us the value of the constant α, but we expect that it will be of order unity. In fact, the correct equation, from the theory of classical hydrodynamics, is

$$v = \sqrt{\frac{g\lambda}{2\pi}} \tag{1.5}$$

or more simply

$$v = \sqrt{g/k} \tag{1.6}$$

where $k = 2\pi/\lambda$ is the ***wave number***. Since $g = 9.81$ m/s^2, we see that a water wave having a wavelength of 1 m will travel at a speed of $\sqrt{9.81/2\pi} = 1.25$ m/s.

Our equation is valid only in deep water, where the depth $h \gg \lambda$. But a tsunami can have a wavelength that is several hundred km. This is much greater than the typical depth of the ocean, about 5 km. For such long wavelengths, the speed is given by

$$v = \sqrt{gh/2\pi} \tag{1.7}$$

which is about 88 m/s (318 km/hr).

1.3 Reference frames

The positions of a point in space can be described by specifying its distance from a reference point, called the ***origin***, along three independent directions. This requires three numbers, for example, (x, y, z), which are called the ***coordinates*** of the point. Clearly, the coordinates have meaning only with respect to a particular choice of the origin and the reference directions, together called a ***frame of reference***. In a different reference frame, the same point will generally have different coordinates. If the point, or the reference frame, is moving, the coordinates will be functions of time.

A frame of reference that is stationary, or moving with constant velocity, is called an ***inertial frame***. We shall see that inertial frames play a special role in classical mechanics.

In a ***Cartesian*** coordinate system, the reference directions are mutually orthogonal (at right angles to each other) and are independent of the position of the point. We will use this type of frame frequently as it simplifies many calculations. However, there will be situations where a non-Cartesian system, such as spherical polar coordinates, is better suited to the symmetry of the problem.

1.4 Coordinate transformations

Consider two Cartesian coordinate systems, O and O' that have the same origin but are rotated with respect to each other about the z axis by an angle θ, in the direction of the right-hand rule. Then, a point that has coordinates (x, y, z) in the O reference frame will have coordinates

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta, \\ y' &= -x\sin\theta + y\cos\theta, \\ z' &= z, \end{aligned}$$

in the O' frame.

This can be written in matrix form,

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{1.8}$$

or more simply

$$\boldsymbol{x}' = \boldsymbol{R}\boldsymbol{x}. \tag{1.9}$$

This is an example of a ***linear transformation*** between the coordinates $\boldsymbol{x}$ and $\boldsymbol{x}'$. In this case it is a simple rotation but more generally, $\boldsymbol{R}$ can be *any* non-singular (invertible) matrix which can represent a rotation about an arbitrary axis, stretching and possibly reflection of the reference axes.

The above rotation $\boldsymbol{R}(\theta)$ can be undone by rotating again by an angle $-\theta$. Therefore

$$\boldsymbol{x} = \boldsymbol{R}(-\theta)\boldsymbol{x}' = \boldsymbol{R}(-\theta)\boldsymbol{R}(\theta)\boldsymbol{x}. \tag{1.10}$$

Therefore

$$\boldsymbol{R}(-\theta)\boldsymbol{R}(\theta) = \boldsymbol{I}, \tag{1.11}$$

where $\boldsymbol{I}$ is the identity matrix that has ones along the diagonal and zeros elsewhere. This means that $\boldsymbol{R}(-\theta)$ is the ***inverse*** of $\boldsymbol{R}(\theta)$,

$$\boldsymbol{R}(-\theta) = \boldsymbol{R}^{-1}(\theta) \tag{1.12}$$

Recalling that $\sin(-\theta) = -\sin\theta$ and $\cos(-\theta) = \cos\theta$, we see that $\boldsymbol{R}(-\theta)$ is equal to the ***transpose*** of $\boldsymbol{R}(\theta)$. Therefore, the rotation matrix satisfies

$$\boldsymbol{R}^T = \boldsymbol{R}^{-1} \tag{1.13}$$

Matrices satisfying this condition are called ***orthogonal matrices***. They form a group called the ***orthogonal group***. A rotation about any axis can be represented by an orthogonal matrix.

1.5 Scalars

Some physical quantities can be represented by a single number, or function of position. An example is the temperature at each point inside a room. If we rotate our reference frame, the temperature does not change, even though the coordinates of the point do. In other words,

$$T'(\boldsymbol{x}') = T(\boldsymbol{x}). \tag{1.14}$$

Quantities or functions that behave this way are called ***scalars***.

1.6 Vectors

Other quantities, such as force or velocity require both a magnitude and a direction to specify them. These are ***vectors*** and can be represented geometrically by arrows. We represent vectors using bold-face type, such at $\boldsymbol{A}$, and the length of the vector by regular type and also by enclosing the vector between vertical lines,

$$A = |\boldsymbol{A}|. \tag{1.15}$$

1.7 Vector equality

Two vectors are said to be equal if they have the same length and same direction. It does not matter where they are located. One is free to move vectors around in order to compare them.

1.8 Vector addition and scalar multiplication

Vector addition can be defined geometrically as the result of putting the tail of one vector on the head of the other. Their sum is a vector extending from the tail of the second to the head of the first.

Multiplying a vector by a number (a scalar) changes the length of the vector but not its direction. For example $2\boldsymbol{V}$ is a vector that points in the same direction as $\boldsymbol{V}$ but is twice as long. $-\boldsymbol{V} = -1\boldsymbol{V}$ is a vector that has the sem length as $\boldsymbol{V}$ but points in the *opposite* direction.

1.9 Unit vectors and components

Unit vectors have unit length. Place three unit vectors at the origin of a Cartesian reference frame and orient them to point along the x, y and z directions. Call them $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ respectively. Then an arbitrary vector V can be written as a sum

$$\boldsymbol{V} = V_x\boldsymbol{i} + V_y\boldsymbol{j} + V_z\boldsymbol{k}. \tag{1.16}$$

The numbers V_x, V_y, V_z are called ***components*** of the vector $\boldsymbol{V}$. In a Cartesian frame, these are just the coordinates of the point at the tip of the vector when its tail is at the origin. In this reference frame, we can represent the vector by a column matrix of components

$$\boldsymbol{V} = \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}. \tag{1.17}$$

If we now rotate the reference frame (without rotating the vector $\boldsymbol{V}$), the orientation of the unit vectors will change. Therefore in the rotated frame, the components of $\boldsymbol{V}$ will be different. They will now correspond to the coordinates of the tip of the vector in the rotated frame. Looking back at our discussion of coordinate transformations, we see that the components are related by the *same* matrix $\boldsymbol{R}$ that relates the coordinates of the two frames,

$$\boldsymbol{V}' = \boldsymbol{R}\boldsymbol{V}. \tag{1.18}$$

We shall often just denote a vector by its components as a row vector, $bsA = (A_x, A_y, A_z)$ with the understanding that we actually mean a column vector.

As an example, consider the ***position vector*** which extends from the origin to some point P. In a Cartesian frame, the coordinates of this point are (x, y, z). These are the same as the components of the position vector, since we can write this vector in the form

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}. \tag{1.19}$$

Vector addition and scalar multiplication can now be defined algebraically, by means of components.

$$a\boldsymbol{A} = (aA_x, aA_y, aA_z), \tag{1.20}$$
$$\boldsymbol{A} + \boldsymbol{B} = (A_x + B_x, A_y + B_y, A_z + B_z). \tag{1.21}$$

1.10 Scalar product

Two kinds of multiplication are defined for vectors. The simplest is the ***scalar product***, also called the ***dot product***. In terms of components, it is defined by

$$\boldsymbol{A} \cdot \boldsymbol{B} = A_xB_x + A_yB_y + A_zB_z. \tag{1.22}$$

The dot product has a coordinate-free geometrical interpretation. To see this, choose a frame in which $\boldsymbol{A}$ lies along the x axis and $\boldsymbol{B}$ is in the $x - y$ plane. Then

$$\begin{aligned} \boldsymbol{A} &= A_x\boldsymbol{i} = A\boldsymbol{i}, \\ \boldsymbol{B} &= B_x\boldsymbol{i} + B_y\boldsymbol{j} = B\cos\theta\,\boldsymbol{i} + B\sin\theta\,\boldsymbol{j}, \end{aligned} \tag{1.23}$$

where θ is the angle between the two vectors. Thus

$$\boldsymbol{A}\cdot\boldsymbol{B} = AB\cos\theta. \tag{1.24}$$

From this we see that $A^2 = \boldsymbol{A}\cdot\boldsymbol{A} = A_x^2 + A_y^2 + A_z^2$.

It follows that if vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ are perpendicular ($\theta = \pi/2$), then $\boldsymbol{A}\cdot\boldsymbol{B} = 0$ and vice versa. We then say that they are ***orthogonal*** vectors.

1.11 Vector product

Another kind of multiplication is the ***vector product***, also called the ***cross product***, $\boldsymbol{A}\times\boldsymbol{B} = \boldsymbol{C}$. This product gives a new vector defined by,

$$\boldsymbol{A}\times\boldsymbol{B} = (A_yB_z - A_zB_y)\boldsymbol{i} + (A_zB_x. - A_xB_z)\boldsymbol{j} + (A_xB_y - A_yB_x)\boldsymbol{k}. \tag{1.25}$$

From this we see that the product is ***anticommutative***

$$\boldsymbol{A}\times\boldsymbol{B} = -\boldsymbol{B}\times\boldsymbol{A}, \tag{1.26}$$

To find the geometrical interpretation of the cross product, take the dot product with $\boldsymbol{A}$,

$$\boldsymbol{A}\cdot(\boldsymbol{A}\times\boldsymbol{B}) = A_x(A_yB_z - A_zB_y) + A_y(A_zB_x. - A_xB_z) + A_z(A_xB_y - A_yB_x) = 0, \tag{1.27}$$

and similarly for the dot product with $\boldsymbol{B}$. Therefore, $\boldsymbol{A}\times\boldsymbol{B}$ is a vector that is perpendicular to both $\boldsymbol{A}$ and $\boldsymbol{B}$.

To find the magnitude of this vector, chose a frame in which $\boldsymbol{A} = A\boldsymbol{i}$ and $\boldsymbol{B} = B\cos\theta\,\boldsymbol{i} + B\sin\theta\,\boldsymbol{j}$, as before. Putting this into the definition we find

$$\boldsymbol{A}\times\boldsymbol{B} = AB\sin\theta\,\boldsymbol{k}. \tag{1.28}$$

We see that the magnitude is $AB\sin\theta$ and the direction is given by the right-hand rule.

1.12 Vector identities

It is straightforward to prove the following useful identities,

$$\boldsymbol{A}\cdot(\boldsymbol{B}+\boldsymbol{C}) = \boldsymbol{A}\cdot\boldsymbol{B} + \boldsymbol{A}\cdot\boldsymbol{C}, \tag{1.29}$$
$$\boldsymbol{A}\times(\boldsymbol{B}+\boldsymbol{C}) = \boldsymbol{A}\times\boldsymbol{B} + \boldsymbol{A}\times\boldsymbol{C}, \tag{1.30}$$
$$a(\boldsymbol{A}\cdot\boldsymbol{B}) = \boldsymbol{aA}\cdot\boldsymbol{B} = \boldsymbol{A}\cdot\boldsymbol{aB}, \tag{1.31}$$
$$a(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{aA}\times\boldsymbol{B} = \boldsymbol{A}\times\boldsymbol{aB}, \tag{1.32}$$
$$\boldsymbol{A}\cdot\boldsymbol{B}\times\boldsymbol{C} = \boldsymbol{B}\cdot\boldsymbol{C}\times\boldsymbol{A} = \boldsymbol{C}\cdot\boldsymbol{A}\times\boldsymbol{B}, \tag{1.33}$$
$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = (\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{C} \tag{1.34}$$

1.13 The derivative of a vector

Consider a vector $V(t)$ that is changing with time. The time derivative of the vector is defined in the usual manner,

$$\frac{d\boldsymbol{V}}{dt} = \lim_{\Delta t \to 0} \frac{\boldsymbol{V}(t+\Delta t) - \boldsymbol{V}(t)}{\Delta t}. \tag{1.35}$$

This derivative of $\boldsymbol{V}$ is a new vector, which generally does not point in the same direction, or have the same length, as $\boldsymbol{V}$.

For example, consider a point $P(t)$, that is moving with time. The position of this point at time t is described by the position vector $\boldsymbol{r}(t)$. The ***velocity*** of the point is defined as the time derivative of the position vector.

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt}. \tag{1.36}$$

In a Cartesian coordinate system, $\boldsymbol{r}$ can be written in the form given by Eqn (1.19). As long as the frame is not rotating, the basis vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ do not change with time. Therefore,

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \frac{dx}{dt}\boldsymbol{i} + \frac{dy}{dt}\boldsymbol{j} + \frac{dz}{dt}\boldsymbol{k}. \tag{1.37}$$

Time derivatives occur frequently, so it is convenient to use a simpler notation. A dot above a variable will denote a time derivative, so the previous result can be written more simply as

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k}. \tag{1.38}$$

A second time derivative will be denoted with two dots, and so on. The ***acceleration*** is defined as the time derivative of the velocity. Thus

$$\boldsymbol{a} = \dot{\boldsymbol{v}} = \ddot{x}\boldsymbol{i} + \ddot{y}\boldsymbol{j} + \ddot{z}\boldsymbol{k}. \tag{1.39}$$

1.14 Circular motion

As an example, consider a particle moving uniformly in a circle in the $x - y$ plane, with radius r. The angle θ increases with time t according to

$$\theta = \omega t, \tag{1.40}$$

where ω, the ***angular frequency*** is a constant. The position vector then has the form

$$\boldsymbol{r}(t) = \cos(\omega t)\,\boldsymbol{i} + \sin(\omega t)\,\boldsymbol{j},. \tag{1.41}$$

We see that the point returns to the same position after a time $T = 2\pi/\omega$, which is the ***period*** of the motion.

Let's now compute the velocity and acceleration of the particle. Differentiating, we find

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = -\omega\sin(\omega t)\,\boldsymbol{i} + \omega\cos(\omega t)\,\boldsymbol{j}, \tag{1.42}$$

$$\boldsymbol{a} = \dot{\boldsymbol{v}} = -\omega^2\cos(\omega t)\,\boldsymbol{i} - \omega^2\sin(\omega t)\,\boldsymbol{j}, \tag{1.43}$$

Comparing Eqns (1.43) and (1.41) we see that

$$\boldsymbol{a} = -\omega^2 \boldsymbol{r} \tag{1.44}$$

so the acceleration is directed inward, directly towards the centre of the circle.

As expected, the velocity vector is perpendicular to the position vector, and therefore tangent to the circle, which can be seen by taking the dot product of the two vectors,

$$\boldsymbol{v} \cdot \boldsymbol{r} = -\omega^2 \sin(\omega t)\cos(\omega t) + \omega^2 \cos(\omega t)\sin(\omega t) = 0. \tag{1.45}$$

1.15 Polar coordinates

We can also analyze the case of circular motion using ***polar coordinates*** (r, θ) in two dimensions.

As Figure 1.11.1 shows, the coordinate r is the length of the position vector and θ is the angle that it makes with the x axis. Each of these two coordinates is associated with a unit vector. The unit vector points in the direction that the point P moves when the corresponding coordinate increases. So, the unit vector $\boldsymbol{e}_r$ associated with r points in the radial direction, away from the origin.

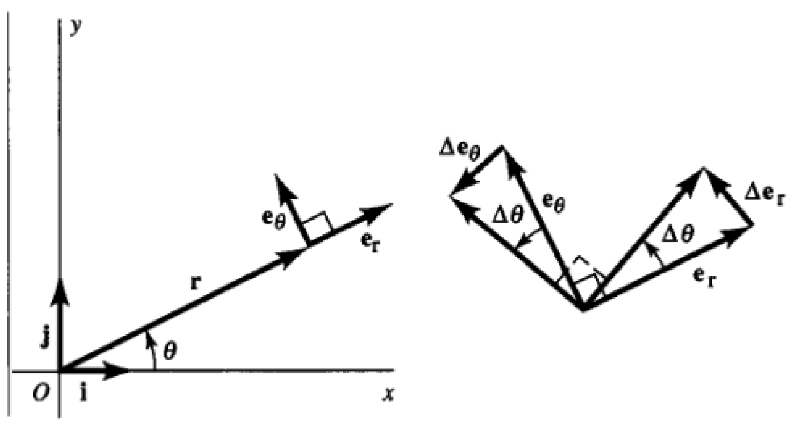

Figure 1.11.1 Unit vectors for plane polar coordinates.

If θ increases, the point P moves perpendicular to $\boldsymbol{r}$, in the direction shown. Thus, the unit vector $\boldsymbol{e}_\theta$ is perpendicular to $\boldsymbol{r}$ and to $\boldsymbol{e}_r$.

Note that the directions of these unit vectors depend on the location of the point P. If this point moves, they will both change their direction in general. We see that for polar coordinates, the unit vectors are *not* constant, but in general will be functions of time.

We can find the derivatives of these unit vectors by letting θ increase by a small amount $\Delta\theta$. Referring to the figure, we see that as $\delta\theta \to 0$ the change in the two vectors is given by

$$\Delta\boldsymbol{e}_r = \Delta\theta\, \boldsymbol{e}_\theta, \tag{1.46}$$
$$\Delta\boldsymbol{e}_\theta = -\Delta\theta\, \boldsymbol{e}_r. \tag{1.47}$$

Dividing by $\Delta\theta$ and taking the limit $\Delta\theta \to 0$, we get

$$\frac{d\boldsymbol{e}_r}{d\theta} = \boldsymbol{e}_\theta, \tag{1.48}$$

$$\frac{d\boldsymbol{e}_\theta}{d\theta} = -\boldsymbol{e}_r. \tag{1.49}$$

Let's now calculate the velocity and acceleration of a particle in circular motion, as before. The position vector is

$$\boldsymbol{r} = r\,\boldsymbol{e}_r \tag{1.50}$$

To get the velocity, we take the derivative with respect to time. The only thing that changes is $\boldsymbol{e}_r$. Using the chain rule, we find

$$\boldsymbol{v} = r\frac{d\boldsymbol{e}_r}{dt} = r\frac{d\theta}{dt}\frac{d\boldsymbol{e}_r}{d\theta} = r\omega\,\boldsymbol{e}_\theta. \tag{1.51}$$

Similarly, we find the acceleration

$$\boldsymbol{a} = r\omega\frac{d\boldsymbol{e}_\theta}{dt} = -r\omega^2\,\boldsymbol{e}_r, \tag{1.52}$$

and we see that $\boldsymbol{a} = -\omega^2\boldsymbol{r}$ as before.

1.16 General motion in three dimensions

In general, all coordinates of an object will vary with time, and with the exception of a non-rotating Cartesian frame, the unit vectors will vary too. To find the velocity and acceleration, we must take the derivatives of all time-varying quantities, using the chain rule.

In a Cartesian frame, the result is simple,

$$\boldsymbol{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}, \tag{1.53}$$

$$\boldsymbol{v} = \dot{x}\,\boldsymbol{i} + \dot{y}\,\boldsymbol{j} + \dot{z}\,\boldsymbol{k}, \tag{1.54}$$

$$\boldsymbol{a} = \ddot{x}\,\boldsymbol{i} + \ddot{y}\,\boldsymbol{j} + \ddot{z}\,\boldsymbol{k}. \tag{1.55}$$

In a cylindrical coordinates, (R, ϕ, z), we must also differentiate the unit vectors,

$$\boldsymbol{r} = R\,\boldsymbol{e}_R + z\,\boldsymbol{e}_z, \tag{1.56}$$

$$\boldsymbol{v} = \dot{R}\,\boldsymbol{e}_R + R\dot{\phi}\,\boldsymbol{e}_\phi + \dot{z}\,\boldsymbol{e}_z, \tag{1.57}$$

$$\boldsymbol{a} = (\ddot{R} - R\dot{\phi}^2)\,\boldsymbol{e}_R + (2\dot{R}\dot{\phi} + R\ddot{\phi})\,\boldsymbol{e}_\phi + \ddot{z}\,\boldsymbol{e}_z, \tag{1.58}$$

$$\tag{1.59}$$

In a spherical coordinates, (r, θ, ϕ), the result is

$$\boldsymbol{r} = r\,\boldsymbol{e}_r, \tag{1.60}$$

$$\boldsymbol{v} = \dot{r}\,\boldsymbol{e}_r + r\dot{\phi}\sin\theta\,\boldsymbol{e}_\phi + r\dot{\theta}\,\boldsymbol{e}_\theta, \tag{1.61}$$

$$\boldsymbol{a} = (\ddot{r} - r\dot{\phi}^2\sin^2\theta - r\dot{\theta}^2)\,\boldsymbol{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\,\boldsymbol{e}_\theta \tag{1.62}$$

$$+ (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)\,\boldsymbol{e}_\phi, \tag{1.63}$$

1.17 Newton's laws

Building upon pioneering experiments and observations by Galileo, Newton proposed three laws of motion:

1. An object at rest, or in uniform motion, continues in that state unless acted on by an external force;
2. The change of motion is proportional to, and in the direction of, the applied force;
3. To every action there is an equal and opposite reaction.

The first law raises philosophical questions. At rest with respect to what? Is a single object, alone in the Universe, at rest? How would one know if it was moving? To address this, Newton's introduced the concept of ***absolute space***. Effectively, this is a frame that all observers agree is at rest. The justification for this concept is solely that Newton's laws correctly predict the observed dynamics of objects, provided that their speeds are much less than the speed of light.

In fact Newton's laws hold in any inertial frame - these are frames moving with constant velocity with respect to the absolute frame. Only in an inertial frame does an object at rest remain at rest, or in a state of uniform motion. Galileo had already reached this conclusion.

Newton's second law is normally written as

$$\boldsymbol{F} = m\boldsymbol{a}, \tag{1.64}$$

where $\boldsymbol{F}$ is the external force, $\boldsymbol{a}$ the acceleration, and the proportionality constant m is called the ***mass*** of the object.

Newton's third law leads to the fundamental concept of conservation of momentum.

1.18 Linear momentum

The (linear) momentum of an object is defined by

$$\boldsymbol{p} = m\boldsymbol{v}. \tag{1.65}$$

Since mass is constant, we can write Newton's second law in the form

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt}. \tag{1.66}$$

Consider an isolated system consisting of two objects, with masses m_1 and m_2, connected by a spring that is pushing them both apart. Here isolated means not acted upon by any external forces. Newton's third law requires that

$$m_1\frac{d\boldsymbol{v}_1}{dt} = -m_2\frac{d\boldsymbol{v}_2}{dt}, \tag{1.67}$$

which we can rewrite as

$$\frac{d}{dt}(\boldsymbol{p}_1 + \boldsymbol{p}_2) = 0 \tag{1.68}$$

This implies that the total momentum

$$\boldsymbol{p}_1 + \boldsymbol{p}_2 = constant \tag{1.69}$$

This is easily extended to any number of objects, and any kind of forces (electrostatic, gravity, etc). We thus arrive at the law of ***conservation of linear momentum***:

The total linear momentum of an isolated system is conserved.

In fact the law of conservation of momentum is more fundamental than Newton's laws and applies even when Newton's laws fail.

2 Motion in one dimension

We often need to solve problems in which a force acts on an object and we wish to determine its subsequent position and velocity as a function of time. This involves solving the equation of motion

$$m\ddot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t). \tag{2.1}$$

This is a second order vector differential equation. The solution will depend on the initial conditions, namely the initial position and velocity of the object.

To illustrate this, we first consider simple problems in which the object is constrained to move in one dimension. In this case the motion can be described by a single scalar parameter such as x, or θ, so a vector equation is not needed.

2.1 Constant force

This is the simplest situation, where the equation of motion is

$$\ddot{x}(t) = \frac{F}{m} \equiv a. \tag{2.2}$$

The solution is found by integrating twice with respect to time,

$$\begin{aligned} v = \dot{x} &= \int \ddot{x} dt = \int a dt = at + v_0, \\ x &= \int v dt = \int (at + v_0) dt = \frac{1}{2}at^2 + v_0 t + x_0. \end{aligned} \tag{2.3}$$

Here v_0 and x_0 are constants of integration, determined by the initial conditions. We see that they correspond to the initial velocity and position, respectively, $v_0 = \dot{x}(0)$, $x_0 = x(0)$.

By eliminating t, one gets the familiar relation between distance and velocity for an object undergoing constant acceleration,

$$v^2 - v_0^2 = 2a(x - x_0). \tag{2.4}$$

2.2 Energy

More interesting is the case when the force depends on position (but not on velocity or time),

$$m\ddot{x}(t) = F(x) \tag{2.5}$$

In this case we can eliminate time by writing

$$\ddot{x} = \frac{dx}{dt}\frac{d\dot{x}}{dx} = v\frac{dv}{dx} \tag{2.6}$$

so

$$F(x) = mv\frac{dv}{dx} = \frac{m}{2}\frac{dv^2}{dx} = \frac{dT}{dx}, \tag{2.7}$$

where

$$T = \frac{1}{2}mv^2 \tag{2.8}$$

is the ***kinetic energy*** of the particle. We can now integrate with respect to x,

$$W = \int_{x_0}^{x} F(x)dx = T - T_0. \tag{2.9}$$

W is the work done by the force as the particle moves from x_0 to x. We see that it is equal to the change in kinetic energy of the object.

To proceed further, let suppose that the force can be written as the derivative of some function $V(x)$,

$$F(x) = -\frac{dV}{dx}. \tag{2.10}$$

This is always true in one dimension, as long as the force is a smooth function of x. Then, the work done is just

$$W = -\int_{x_0}^{x} \frac{dV}{dx}dx = -\int_{V(x_0)}^{V(x)} dV = V(x_0) - V(x). \tag{2.11}$$

$V(x)$ is called the ***potential energy***.

Combining this with the previous expression for W we find that

$$T - T_0 = V(x_0) - V(x) \tag{2.12}$$

so

$$T + V(x) = T_0 + V(x_0) \equiv E. \tag{2.13}$$

E is a constant, since T_0 and $V(x_0)$ do not depend on time. It is called the ***total energy***. We see that as the object moves, the potential and kinetic energies may change, but the total energy remains constant.

The extension to multiple objects is straightforward. We add the kinetic energies of all the objects to get the total kinetic energy, and similarly to get the total potential energy. This leads to the law of ***conservation of energy***.

If the force is not an explicit function of time, the total energy of a system is conserved.

We can now find the velocity of the object as a function of its position. Start with

$$T + V(x) = \frac{m}{2}v^2 + V(x) = E, \tag{2.14}$$

and now solve for v,

$$v(x) = \sqrt{\frac{2}{m}\left[E - V(x)\right]}. \tag{2.15}$$

From this we see that there is no solution if $V(x) > E$. The object is confined to regions that have $V(x) \leqslant E$, Also, if $V(x) = E$, $v = 0$. The object's velocity is zero at positions where the potential energy equals the total energy. These positions are called ***turning points***, because the object reverses direction when it reaches these points.

One can now a relation between the position of the object and time by integrating the velocity,

$$t = \int dt = \int \frac{dx}{v} = \sqrt{\frac{m}{2}} \int \frac{dx}{\sqrt{E - V(x)}}. \tag{2.16}$$

2.3 Velocity-dependent forces

Suppose now that the force depends on the velocity of the particle. Examples are the drag that an object experiences when moving through a fluid, and the Lorentz force on a charged particle moving in a magnetic field. The equation of motion is now

$$F(v) = m\frac{dv}{dt} = mv\frac{dv}{dx}. \tag{2.17}$$

which has solutions

$$t = m\int \frac{dv}{F(v)}, \tag{2.18}$$

$$x = m\int \frac{vdv}{F(v)}. \tag{2.19}$$

For example, the force on a sphere moving through a fluid, including air, can be approximated by the equation

$$F(v) = -v(c_1 + c_2|v|). \tag{2.20}$$

The first term is a linear relationship in which the force is proportional to the velocity. It is called ***Stokes law*** after George Stokes, who first identified it in 1851. The coefficient c_1 is proportional to the viscosity of the fluid and the diameter of the sphere.

The second term is quadratic in the velocity. It represents a drag force caused by momentum transfer from molecules striking the sphere. The change in momentum is proportional to velocity, and so is the rate at which the sphere encounters molecules.

For a sphere of diameter D moving through air,

$$c_1 = 1.55 \times 10^{-4} D,$$
$$c_2 = 0.22 D^2, \tag{2.21}$$

in SI units. These two coefficients are equal for $D = 0.0007$ m $= 0.7$ mm. For objects that are smaller than this, the linear term will dominate. For objects larger than this the quadratic term dominates.

To get an idea of behaviour consider the two limiting cases of linear drag ($c_2|v| \ll c_1$) and quadratic drag ($c_2|v| \gg c_1$). We take the initial velocity to be v_0 at $t = x = 0$.

For the linear case we have

$$t = -m\int_{v_0}^{v} \frac{dv}{c_1 v} = -\frac{m}{c_1}(\ln v - \ln v_0) = -\frac{m}{c_1}\ln\left(\frac{v}{v_0}\right), \tag{2.22}$$

$$x = -m\int_{v_0}^{v} \frac{dv}{c_1} = -\frac{m}{c_1}(v - v_0). \tag{2.23}$$

The first equation can be solved for v by multiplying by $-c_1/m$ and taking the exponential. This gives

$$v = v_0 e^{-c_1 t/m} \tag{2.24}$$

$$x = \frac{mv_0}{c_1}\left(1 - e^{-c_1 t/m}\right). \tag{2.25}$$

We see that the velocity decreases exponentially with a characteristic time $\tau_1 = m/c_1$. In the limit of infinite time, the distance traveled is $v_0\tau_1$.

In the quadratic case, assuming $v > 0$, we have

$$t = -m\int_{v_0}^{v} \frac{dv}{c_2 v^2} = \frac{m}{c_2}\left(\frac{1}{v} - \frac{1}{v_0}\right), \tag{2.26}$$

$$x = -m\int_{v_0}^{v} \frac{dv}{c_2 v} = -\frac{m}{c_2}\ln\left(\frac{v}{v_0}\right). \tag{2.27}$$

Solving the first equation for v we find

$$v = \frac{v_0}{1 + t/\tau_2}. \tag{2.28}$$

where the characteristic time is now $\tau_2 = m/c_2 v_0$. Substituting this in the equation for x gives

$$x = v_0\tau_2 \ln(1 + t/\tau_2). \tag{2.29}$$

In this case, the object never stops. This is because a quadratic force is less effective than a linear force at low speed.

2.4 Falling raindrops

Consider a raindrop of diameter D falling vertically. We suppose that the drop is released with initial velocity $v_0 = 0$ at time $t = 0$ and take $x(t)$ to be the distance traveled downward. The force has two parts,

$$F = mg - v(c_1 + c_2|v|). \tag{2.30}$$

where c_1 and c_2 are defined in Eqn. (2.21).

The equation of motion is

$$\frac{dv}{dt} = v\frac{dv}{dx} = g - \frac{v}{m}(c_1 + c_2|v|). \tag{2.31}$$

To simplify this define $a_1 = c_1/mg$ and $a_2 = c_2/mg$

The solution for t, assuming $v > 0$, is

$$t = \frac{1}{g}\int_0^v \frac{dv}{1 - a_1 v - a_2 v^2} \tag{2.32}$$

2.4.1 Linear approximation

If $a_2 = 0$, the integral becomes

$$t = -\frac{1}{ga_1}\ln(1 - a_1 v) \tag{2.33}$$

which has the solution

$$v(t) = \frac{1}{a_1}\left(1 - e^{-t/\tau_1}\right). \tag{2.34}$$

We see that the velocity exponentially approaches a limiting value of $1/a_1$ with a characteristic time $\tau_1 = 1/ga_1$.

2.4.2 General case

To solve the general case, observe that the denominator can be factored, giving

$$\begin{aligned} t &= -\frac{1}{ga_2}\int_0^v \frac{dv}{(v - v_+)(v - v_-)}, \\ &= -\frac{1}{ga_2(v_+ - v_-)}\int_0^v \left(\frac{1}{v - v_+} - \frac{1}{v - v_-}\right) dv, \\ &= -\frac{1}{g\sqrt{a_1^2 + 4a_2}}\left[\ln(v/v_+ - 1) - \ln(v/v_- - 1)\right], \\ &= -\tau \ln\left(\frac{v/v_+ - 1}{v/v_- - 1}\right), \end{aligned} \tag{2.35}$$

where

$$v_+ = \frac{1}{2a_2}\left(-a_1 + \sqrt{a_1^2 + 4a_2}\right), \tag{2.36}$$

$$v_- = \frac{1}{2a_2}\left(-a_1 - \sqrt{a_1^2 + 4a_2}\right), \tag{2.37}$$

$$\tau = \frac{1}{g\sqrt{a_1^2 + 4a_2}}. \tag{2.38}$$

Solving this for v, we get

$$\begin{aligned} v/v_+ - 1 &= (v/v_- - 1)e^{-t/\tau} \\ v - v_+ &= (v - v_-)\frac{v_+}{v_-}e^{-t/\tau} \\ v\left(1 - \frac{v_+}{v_-}e^{-t/\tau}\right) &= v_+\left(1 - e^{-t/\tau}\right) \end{aligned} \tag{2.39}$$

so

$$v = v_+\frac{1 - e^{-t/\tau}}{1 - (v_+/v_-)e^{-t/\tau}}. \tag{2.40}$$

We see that as $t \to \infty$, the velocity approaches the limit $v \to v_+$, with a characteristic time τ. This speed is called the *terminal velocity*

$$\begin{aligned} v_+ &= \frac{1}{2a_2}\left(-a_1 + \sqrt{a_1^2 + 4a_2}\right) = \frac{c_1}{2c_2}\left(\sqrt{1 + 4mgc_2/c_1^2} - 1\right), \\ &= \frac{0.00035}{D}\left(\sqrt{1 + 3.59 \times 10^8 m} - 1\right) \simeq 6.6\frac{\sqrt{m}}{D}. \end{aligned} \tag{2.41}$$

Since water has a density $\rho = 1000$ kg/m^3, a 2 mm diameter raindrop will have a mass of

$$m = \rho V = \frac{4\pi R^3 \rho}{3} = \frac{4\pi(0.001)^3(1000)}{3} = 4.19 \times 10^{-6} \text{ kg}, \tag{2.42}$$

and a terminal velocity of

$$v_+ = 6.6\frac{\sqrt{4.19 \times 10^{-6}}}{0.002} = 6.8 \text{ m/s}. \tag{2.43}$$

A larger object, such as a spherical skydiver of mass 70 kg and diameter 0.5 m, would reach a terminal velocity of 110 m/s (about 400 km/hr).

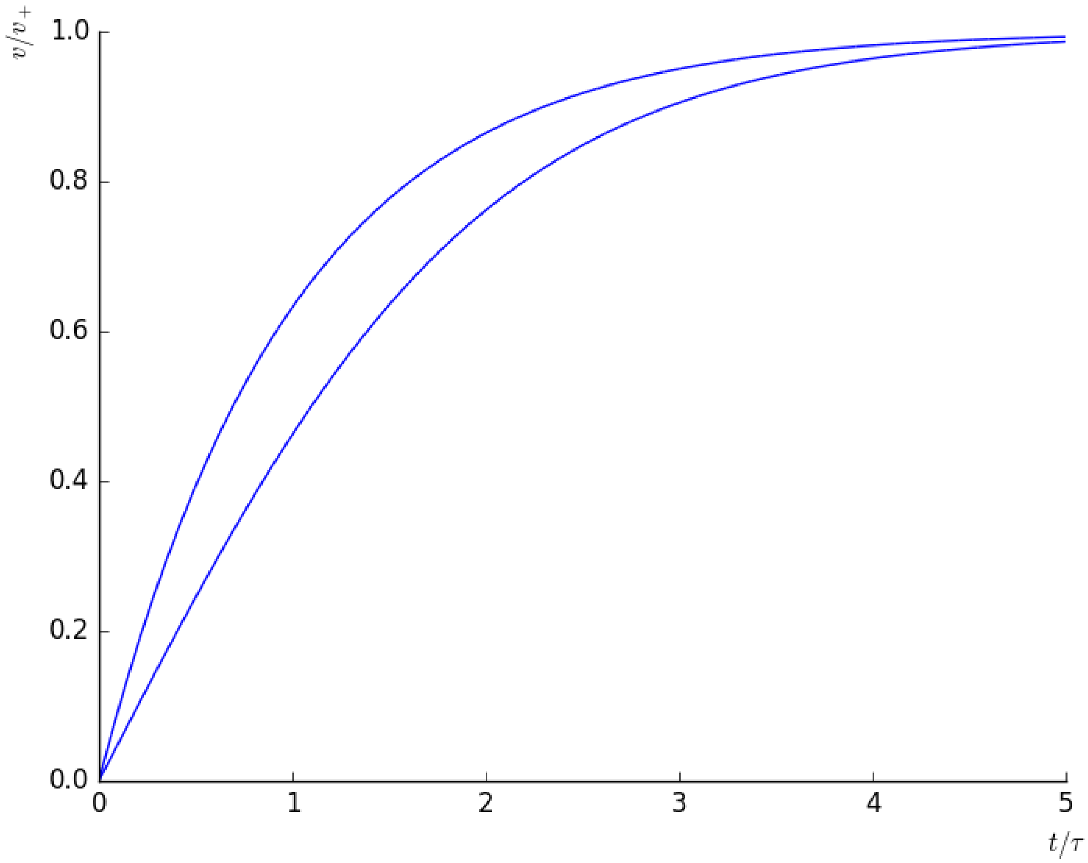

Figure 2.1: Raindrop velocity vs. time. The upper curve shows the limit of linear drag, the lower curve shows the limit of pure quadratic drag.

3 Oscillatory motion

Lets return now to the case when the force is a function of position $F = F(x)$. In that case the equation of motion is given by

$$m\ddot{x} = -F(x) = -\frac{dV(x)}{dx}, \tag{3.1}$$

where $V(x)$ is the potential energy function.

Suppose that $V(x)$ has a minimum at some value of x. Call this value x_0. Let $u = x - x_0$ be the displacement from this minimum position. Since x_0 is a constant, $du = dx$, so $dV/dx = dV/du$

At the minimum, $u = 0$. Also, the derivative $dV/du = 0$ at this point.

If V(u) is a smooth function, it can be approximated by a polynomial, as follows,

$$V(u) = a_0 + a_1 u + a_2 u^2 + a_3 u^3 + \cdots, \tag{3.2}$$

where $a_0, a_1, a_2, a_3, \ldots$ are constants. In terms of the variable u, the force is now

$$F(u) = -\frac{dV}{dx} = -\frac{dV}{du} = -(a_1 + 2a_2 u + 3a_3 u^2 + \cdots) \tag{3.3}$$

Because the derivative of V is zero when $u = 0$, we must have $a_1 = 0$.

So the equation of motion is now

$$m\ddot{u} = -2a_2 u - 3a_3 u^2 - \cdots. \tag{3.4}$$

If the displacement u is small, the higher-order terms $a_3 u^2$, $a_4 u^3$, etc will be small compared to the linear term $a_2 u$. In the limit of small displacements, they can be neglected. (This amounts to approximating the potential function with a parabola.) To simplify what follows define the ***spring constant*** constant $k = 2a_2$.

We thus obtain a second-order linear differential equation

$$\ddot{u} = -\frac{k}{m} u \tag{3.5}$$

which describes a simple ***harmonic oscillator***.

It is easy to verify that a solution is

$$u = A \sin(\omega_0 t + \phi_0), \tag{3.6}$$

where A and ϕ_0 are arbitrary constants and

$$\omega_0 = \sqrt{\frac{k}{m}}. \tag{3.7}$$

This represents a sinusoidal oscillation, with amplitude A and period $T_0 = 2\pi/\omega_0$. The parameter ω_0 is called the ***angular frequency***. The constant ϕ_0 describes the phase of the oscillation at $t = 0$.

The velocity is found by differentiating this solution

$$\dot{u} = A\omega_0 \cos(\omega_0 t + \phi_0), \tag{3.8}$$

If the values of u and $v = \dot{u}$ are specified at some time, say $t = 0$, then we can determine A and ϕ_0 by solving the two equations

$$u_0 = u(0) = A\sin(\phi_0), \tag{3.9}$$
$$v_0 = v(0) = A\omega_0 \cos(\phi_0) \tag{3.10}$$

so

$$\phi_0 = \arctan\left(\frac{\omega_0 u_0}{v_0}\right) \tag{3.11}$$
$$A^2 = u_0^2 + \frac{v_0^2}{\omega_0^2}. \tag{3.12}$$

3.1 Vibrating diatomic molecule

As an example of oscillatory motion, consider a diatomic molecule consisting of two atoms of masses m_1 and m_2 separated by a distance d. Let the position vectors of the two atoms, at time t, be $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. The separation between the two atoms can be represented by the vector $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$.

We chose an inertial reference frame in which the total momentum $\boldsymbol{p} = \boldsymbol{p}_1 + \boldsymbol{p}_2 = 0$. In this frame,

$$m_1\dot{\boldsymbol{r}}_1 + m_2\dot{\boldsymbol{r}}_1 = 0, \tag{3.13}$$

Which can be written as

$$\frac{d}{dt}\left(m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2\right) = 0, \tag{3.14}$$

so

$$m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_1 = const, \tag{3.15}$$

We chose the location the origin so that the constant is zero. This is called the ***centre-of-mass frame***. In this frame,

$$m_1\boldsymbol{r}_1 = -m_2\boldsymbol{r}_1. \tag{3.16}$$

The potential energy of the system is a function of the separation r between the two atoms. It can be described by the *Morse potential*,

$$V(x) = V_0[1 - e^{-x/\delta}]^2 - V_0 = -2V_0 e^{-x/\delta} + V_0 e^{-2x/\delta}. \tag{3.17}$$

where $x = r - r_0$ is the displacement from the equilibrium separation r_0.

If the molecule is stretched or compressed, the first atom feels a force $\boldsymbol{F}_1(r)$ directed towards the equilibrium position. The second atom feels an equal and opposite force $\boldsymbol{F}_2 = -\boldsymbol{F}_1$ (imagine a spring joining the two atoms).

In vector form, the equations of motion for the two atoms are

$$\ddot{\boldsymbol{r}}_1 = \frac{1}{m_1}\boldsymbol{F}_1(r) \tag{3.18}$$

$$\ddot{\boldsymbol{r}}_2 = \frac{1}{m_2}\boldsymbol{F}_2(r) \tag{3.19}$$

$$\tag{3.20}$$

If we subtract the second equation from the first we get

$$\ddot{\boldsymbol{r}} = \frac{1}{\mu}\boldsymbol{F}_1(r), \tag{3.21}$$

where μ, defined by

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \tag{3.22}$$

is called the ***reduced mass*** of the system.

We can now take the magnitude of this vector equation and write the result in terms of the displacement u. This gives a single differential equation

$$\ddot{x} = \frac{1}{\mu}F_1(x) = -\frac{1}{\mu}\frac{dV(x)}{dx}. \tag{3.23}$$

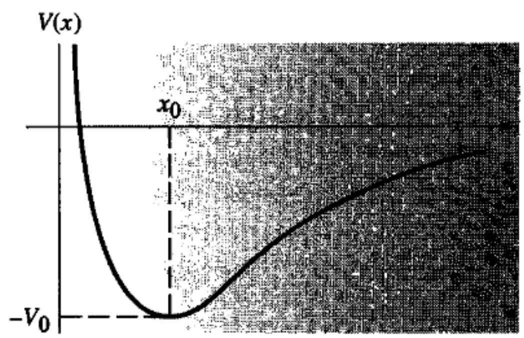

Figure 2.3.2 Potential energy function for a diatomic molecule.

Let's now approximate the Morse potential by a polynomial. To do this we make a ***Taylor expansion*** about the minimum, $x = 0$,

$$V(x) = V(0) + x\left[\frac{dV}{dx}\right]_{x=0} + \frac{x^2}{2!}\left[\frac{d^2V}{dx^2}\right]_{x=0} + \frac{x^3}{3!}\left[\frac{d^3V}{dx^3}\right]_{x=0} + \cdots . \tag{3.24}$$

We know that the second (linear) term must be zero because the potential has a minimum at $x = 0$. Therefore

$$V(x) = V(0) + \frac{1}{2}\left[\frac{d^2V}{dx^2}\right]_{x=0} x^2 + \cdots \tag{3.25}$$

For small displacements, $x \ll \delta$, the higher order terms are small compared to the quadratic term and can be ignored in a first approximation. Our equation of motion reduces to that of a simple harmonic oscillator,

$$\ddot{x} = -\frac{k}{\mu}x = -\omega_0^2 x. \tag{3.26}$$

with spring constant

$$k = \left[\frac{d^2V}{dx^2}\right]_{x=0} = \frac{2V_0}{\delta^2}. \tag{3.27}$$

If excited, the diatomic molecule will vibrate about its equilibrium separation with frequency $\omega_0 = \sqrt{k/\mu}$.

3.2 The rigid pendulum

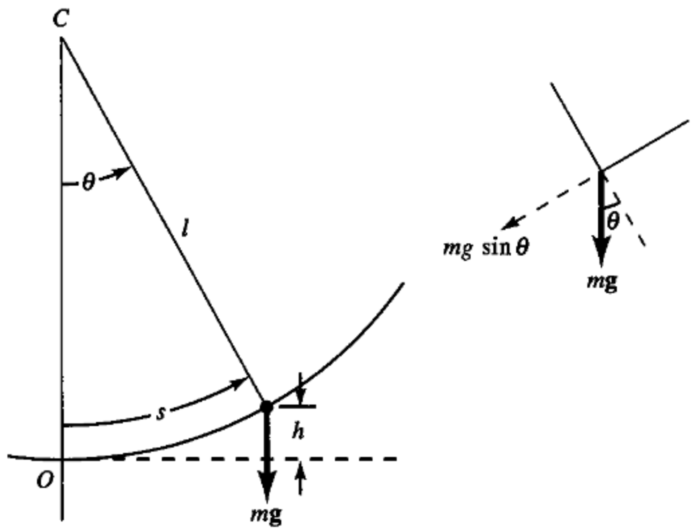

Figure 3.2.6 The simple pendulum.

A simple example of oscillatory motion is the simple pendulum. Here, a mass m is attached to the end of a rigid rod of length l that is free to pivot about its endpoint C. Let θ be the angle made by the rod with the vertical. On one side θ is positive and on the other side it is negative.

It is simplest to use a polar coordinate system, centred at C. In this system, the mass has coordinates (r, θ) with $r = l$. Because the mass is constrained to move only in the θ direction (positive or negative), the only components of velocity and acceleration that we need consider are the θ components.

At any given time, these components will be

$$v_\theta = \frac{d}{dt}(r\theta) = l\dot{\theta}, \tag{3.28}$$

$$a_\theta = \dot{v}_\theta = l\ddot{\theta}. \tag{3.29}$$

The force $\boldsymbol{F}$ is provided by gravity, which acts in the downward vertical direction with magnitude mg. The component of this force in the θ direction is

$$F_\theta = \boldsymbol{e}_\theta \cdot \boldsymbol{F} = -mg\sin\theta \tag{3.30}$$

The equation of motion is therefore

$$ml\ddot{\theta} = -mg\sin\theta \tag{3.31}$$

which simplifies to

$$\ddot{\theta} = -\frac{g}{l}\sin\theta. \tag{3.32}$$

In principle we could solve this exactly using the general methods developed in the previous section. However this leads to integrals that have no analytic solution. A simpler way to proceed is to expand $\sin\theta$ in a Taylor series,

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots \tag{3.33}$$

If the angle θ is small enough, we can just keep the linear term and ignore the higher-order terms. We then have the equation of a harmonic oscillator,

$$\ddot{\theta} = -\frac{g}{l}\theta. \tag{3.34}$$

We see that the pendulum oscillates with an angular frequency

$$\omega_0 = \sqrt{\frac{g}{l}}, \tag{3.35}$$

which is independent of the mass. The period is

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{l}{g}}. \tag{3.36}$$

The effect of the non-linear terms in the expansion of $\sin\theta$ will be considered later.

3.3 Energy of a harmonic oscillator

In a harmonic oscillator, the potential corresponding to the force $-kx$ is

$$V(x) = \frac{k}{2}x^2. \tag{3.37}$$

This is the potential energy of the oscillator. The kinetic energy is $T = m\dot{x}^2/2$ and the sum $T + V = E$ is the total energy, which is constant.

The turning points of the oscillator are the values of x for which $v = \dot{x} = 0$, which correspond to $V = E$.

3.4 Effect of a constant force on a harmonic oscillator

Suppose we have a mass m hanging from a spring. Let x measure the vertical position of the mass (increasing in the downward direction). Then the force on the mass is

$$F = mg - k(x - x_e) \tag{3.38}$$

where x_e is the equilibrium position of the mass if there was no gravitational force.

When the mass is attached, the spring will extend to a new equilibrium x_0 position that results in no net force on the mass. Therefore

$$0 = mg - k(x_0 - x_e), \tag{3.39}$$

so

$$x_0 = x_e + \frac{mg}{k} \tag{3.40}$$

In terms of the displacement from the new equilibrium position, the force is

$$F = mg - k(x - x_e) = -k(x - x_0) \tag{3.41}$$

and the constant force has been absorbed into the change in equilibrium position. *Apart from a change in equilibrium position, a constant force has no effect on the motion of a harmonic oscillator.*

3.5 Damped harmonic oscillator

In our discussion of harmonic motion, we have so far ignored frictional forces. A pendulum, or an object on a spring, will loose energy to frictional or drag forces. As a result, its amplitude of oscillation will gradually decrease. This is called ***damping***.

As an example, consider a mass hanging vertically from a spring and suppose that it is subject to a linear drag force $\boldsymbol{F} = -c\boldsymbol{v}$. Let x denote the displacement from the equilibrium position, the equation of motion will now be

$$m\ddot{x} = -kx - c\dot{x}. \tag{3.42}$$

This can be written more simply as

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0 \tag{3.43}$$

where

$$\omega_0 = \sqrt{\frac{k}{m}} \tag{3.44}$$

is the angular frequency and

$$\gamma = \frac{c}{2m} \tag{3.45}$$

is the ***damping coefficient***.

To solve this equation, let D denote the ***differential operator*** d/dt. The equation of motion can then be written in the form

$$[D^2 + 2\gamma D + \omega_0^2]x = 0, \tag{3.46}$$

which can be factored to give

$$\left[D + \gamma - \sqrt{\gamma^2 - \omega_0^2}\right]\left[D + \gamma + \sqrt{\gamma^2 - \omega_0^2}\right]x = 0, \tag{3.47}$$

This has two solutions, obtained by setting each of the two factors equal to zero. Let

$$q = \sqrt{\gamma^2 - \omega_0^2}. \tag{3.48}$$

The first factor gives us

$$\frac{dx}{dt} + (\gamma - q)x = 0 \tag{3.49}$$

which has the solution

$$x = A_1 e^{-(\gamma - q)t} \tag{3.50}$$

for any constant A_1. The second term is similar but with a different sign, so the general solution will be

$$x = A_1 e^{-(\gamma-q)t} + A_2 e^{-(\gamma+q)t} \tag{3.51}$$

The values of A_1 and A_2 are determined by the initial conditions.

The nature of the solution depends on whether q is real or imaginary. The possibilities are

1. $q > 0$ - overdamping
2. $q = 0$ - critical damping
3. q imaginary - underdamping

3.5.1 overdamping

If $q > 0$ the displacement x returns to 0 smoothly and monotonically, with two different time constants, $\tau_1 = 1/(\gamma - q)$ and $\tau_2 = 1/(\gamma + q)$.

3.6 critical damping

If $q = 0$, both terms in Eqn (3.51) have the same form and can be added, leaving only one free constant. But we need two constants and therefore two independent solutions in order to satisfy general initial conditions (initial position and velocity). The equation of motion is now

$$(D + \gamma)(D + \gamma)x = 0. \tag{3.52}$$

Let $u = (D + \gamma)x$. Then

$$(D + \gamma)u = 0, \tag{3.53}$$

which has the solution

$$u = Ae^{-\gamma t}. \tag{3.54}$$

Therefore,

$$A = e^{\gamma t}u = e^{\gamma t}(D + \gamma)x = D(e^{\gamma t}x). \tag{3.55}$$

This can be integrated to give

$$x = Ate^{-\gamma t} + Be^{-\gamma t} \tag{3.56}$$

which has two independent constants as required.

In this case $\gamma = \omega_0$ and a displacement returns smoothly to zero with a characteristic time of $\tau = 1/\gamma$.

3.7 Underdamping

In this case $\gamma < \omega_0$ so q is imaginary. We can write $q = i\omega_d$ where

$$\omega_d = \sqrt{\omega_0^2 - \gamma^2} \tag{3.57}$$

The general solution now has the form

$$x(t) = e^{-\gamma t}\left(C_1 e^{i\omega_d t} + C_2 e^{-i\omega_d t}\right). \tag{3.58}$$

where C_1 and C_2 are complex constants.

We can convert this result to an expression involving trigonometric functions by means of Euler's equation

$$e^{ix} = \cos x + i \sin x. \tag{3.59}$$

Substitution gives

$$\begin{aligned} x(t) &= e^{-\gamma t}\left\{C_1[\cos(\omega_0 t) + i\sin(\omega_0 t)] + C_2[\cos(\omega_0 t) - i\sin(\omega_0 t)]\right\}, \\ &= e^{-\gamma t}\left[(C_1 + C_1)\cos(\omega_0 t) + i(C_1 - C_2)\sin(\omega_0 t)\right]. \end{aligned} \tag{3.60}$$

The expression on the right hand side involves complex numbers, but x has to be real since it represents a physical displacement. We conclude that $C_1 + C_2 \equiv A_1$ and $i(C_1 - C_2) \equiv A_2$ must be real constants. Our solution is therefore

$$x(t) = e^{-\gamma t}[A_1 \cos(\omega_d t) + A_2 \sin(\omega_d t)]. \tag{3.61}$$

One can use trig relations to convert this to the alternative form

$$x(t) = e^{-\gamma t} A \cos(\omega_d t + \phi_0). \tag{3.62}$$

where the constants are related by $A_1 = A\cos(\phi_0)$ and $A_2 = -A\sin(\phi_0)$.

We see that the motion corresponds to a harmonic oscillator with angular frequency $\omega_d = \sqrt{\omega_0^2 - \gamma^2}$ having an amplitude that decays exponentially with time. This is illustrated in Figure 3.4.3 in Fowles and Cassiday.

3.8 Energy of a damped oscillator.

Since the motion of a damped oscillator decays, its total energy must also decrease. We can easily calculate the rate of energy loss,

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right) = (m\ddot{x} + kx)\dot{x}. \tag{3.63}$$

Now substitute the equation of motion $m\ddot{x} + c\dot{x} + kx = 0$, so $m\ddot{x} + kx = -c\dot{x}$ and we again get

$$\frac{dE}{dt} = -c\dot{x}^2. \tag{3.64}$$

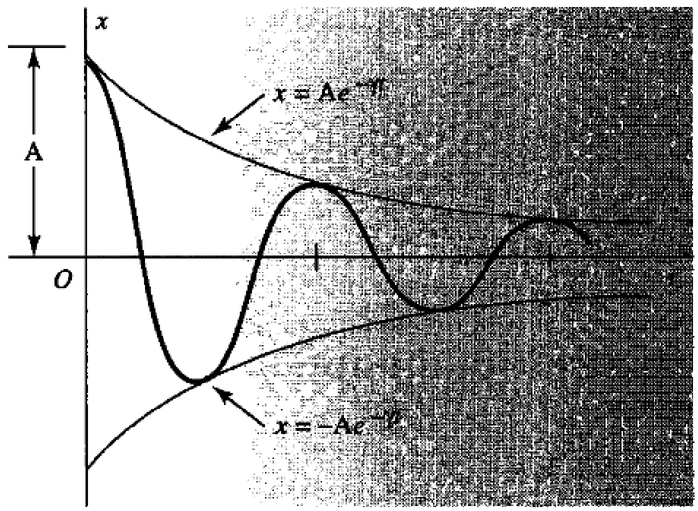

Figure 3.4.3 Graph of displacement versus time for the underdamped harmonic oscillator.

Where does this energy go? It is dissapated (eventually into heat) by the mechanism that produces the damping. The rate of work done by the damping force is

$$\frac{dW}{dt} = \boldsymbol{F} \cdot \boldsymbol{v} = -cv^2. \tag{3.65}$$

which we see equals the rate of energy loss.

Since both the velocity and the displacement of the motion are proportional to $e^{-\gamma t}$, and the kinetic and potential energies are proportional to the squares of these quantities, it follows that the total energy decays exponentially with a characteristic time $\tau = 1/2\gamma$,

$$E = E_0 e^{-2\gamma t} = E_0 e^{-t/\tau} \tag{3.66}$$

3.9 Quality factor

For a weakly-damped harmonic oscillator, the rate of energy loss can be characterized by a dimensionless parameter Q called the ***quality factor***. An oscillator with very slow energy loss has a high Q and vice versa.

This factor is defined as 2π times the ratio of the energy decay time constant τ to period of the oscillator $T = 2\pi/\omega_d$,

$$Q \equiv \frac{2\pi\tau}{T} = \frac{\omega_d}{2\gamma}. \tag{3.67}$$

Values of Q for naturally occurring oscillations range from a few hundred for the Earth (earthquakes) to more than 10^{12} for neutron stars and atomic nuclei (see the Table 3.4.1 in the text book).

3.10 Phase space trajectories

Phase space is the space spanned by the coordinates of position and momentum, or equivalently position and velocity. For one-dimensional motion, phase space has two dimensions, usually taken to be x and v.

As an oscillating system evolves with time, the position and velocity change, tracing out a trajectory in phase space. For a harmonic oscillator, the energy equation

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \tag{3.68}$$

can be cast in the form

$$\frac{x^2}{a^2} + \frac{v^2}{b^2} = 1, \tag{3.69}$$

where $a^2 = 2E/k$ and $b^2 = 2E/m$ are constants. This is the equation of an ellipse in phase space, with a and b being the maximum values of x and v respectively. The axis ratio of the ellipse is

$$\frac{b}{a} = \sqrt{\frac{k}{m}} = \omega_0 \tag{3.70}$$

and the size of the ellipse is proportional to E, the total energy of the system. These curves are illustrated in Figure 3.5.1 in the text book (with the y axis depicting velocity).

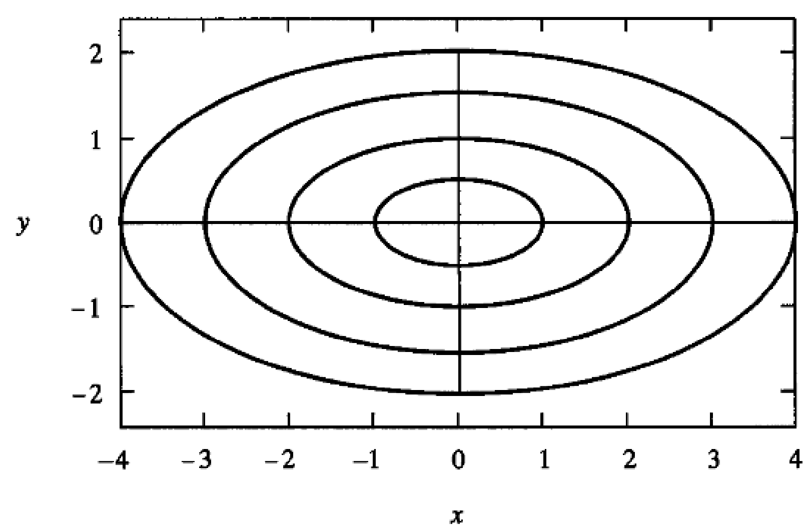

Figure 3.5.1 Phase-space plot for the simple harmonic oscillator ($\omega_0 = 0.5\ \mathrm{s}^{-1}$). No damping force ($\gamma = 0\ \mathrm{s}^{-1}$).

For a damped oscillator, energy is lost to friction, so the trajectory spirals inward and eventually reaches the equilibrium point $x = v = 0$. The shape of the trajectory depends on the degree of damping (see Figure 3.5.3 in Fowles and Cassiday).

3.11 Stability

An ***equilibrium point*** is a point in phase space where the system will remain if not disturbed. An example is the point ($x = 0, v = 0$) for a harmonic oscillator. For example, a mass on a spring that is at rest will remain at rest because there is no net force on the mass.

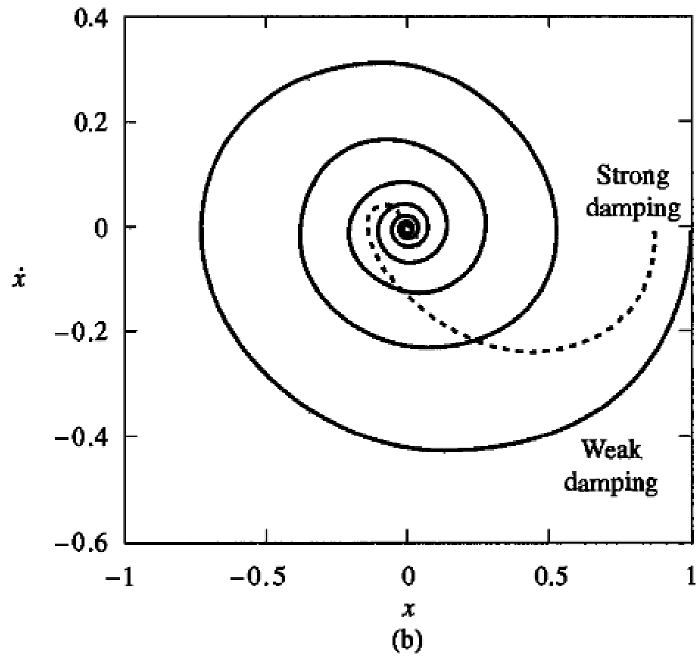

Figure 3.5.3 (a) Modified phase-space plot (see text) for the simple harmonic oscillator. (b) Phase-space plot ($\omega_0 = 0.5\ s^{-1}$). Underdamped case: (1) weak damping ($\gamma = 0.05\ s^{-1}$) and (2) strong damping ($\gamma = 0.25\ s^{-1}$).

An equilibrium point is said to be ***stable*** if the system returns to this point if it is disturbed by a small amount. Suppose for example that we have a mass on a spring, at rest. If we push it a little it will oscillate up and down and gradually come back to rest at the equilibrium point. This happens because a displacement results in a force directed towards the equilibrium point.

This is equivalent to the statement that the potential curves upwards on either side of the equilibrium point. In other words, the second derivative $V''(x)$ is positive here.

An equilibrium point is said to be ***unstable*** if the system moves away from this point if disturbed. This will happen if a displacement results in a force directed away from the equilibrium point, which corresponds to the second derivative of the potential being negative at the equilibrium point.

A simple way to remember this is the following:

Equilibrium points correspond to peaks or valleys (or inflection points) in the potential $V(x)$.

If the point is at the bottom of a valley it is stable.

If the point is at the top of a peak, it is unstable.

3.12 Driven harmonic oscillator

In many situations we are faced with a system that is driven or perturbed by a force that is periodic in time. If the system has a natural frequency of oscillation which is close to the driving frequency, the oscillations can grow to vary large amplitude. This is called ***resonance***.

Consider a harmonic oscillator with natural frequency ω_0 to which is applied a period force $F = F_0 \cos(\omega t)$. The equation of motion is

$$\ddot{x} + \omega_0^2 x = \frac{F_0}{m} \cos(\omega t). \tag{3.71}$$

It is often easier to work with complex exponential functions than with trigonometric functions. From the Euler equation we see that

$$\cos(\omega t) = \text{Re } e^{i\omega t} \tag{3.72}$$

where Re denotes the real part of the complex number. We can omit it and work with the complex numbers directly as long as we remember that at the end we need to take the real part of our equation. In complex form, our equation becomes

$$\ddot{x} + \omega_0^2 x = \frac{F_0}{m} e^{i\omega t}. \tag{3.73}$$

If we wait long enough we expect that the system will reach a steady state in which it oscillates at the driving frequency ω, but with an amplitude and phase to be determined. So we try the expression

$$x = Ce^{i\omega t} \tag{3.74}$$

where C is some complex constant that encodes the amplitude and phase. Substituting this in the equation of motion gives

$$-\omega^2 Ce^{i\omega t} + \omega_0^2 Ce^{i\omega t} = \frac{F_0}{m} e^{i\omega t}. \tag{3.75}$$

Solving this for C we find

$$C = \frac{F_0/m}{\omega_0^2 - \omega^2} \tag{3.76}$$

so

$$x = \text{Re } \frac{F_0/m}{\omega_0^2 - \omega^2} e^{i\omega t} = \frac{F_0/m}{\omega_0^2 - \omega^2} \cos(\omega t). \tag{3.77}$$

We see that if the driving frequency ω is less than the natural frequency ω_0, the oscillation is in phase with the driving force.

As ω approaches ω_0 the amplitude increases, becoming infinite (for an ideal system with no friction or damping) when $\omega = \omega_0$. This is the resonant frequency.

Finally, when $\omega > \omega_0$ the constant C changes sign, which means that the oscillations are now *opposite* to the driving force. In other words, the oscillation is 180° out of phase with the driving force.

3.13 Damped driven harmonic oscillator

Lets now add some damping and see what happens. The equation of motion becomes

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m} e^{i\omega t}. \tag{3.78}$$

Again we try the solution

$$x = Ce^{i\omega t} \tag{3.79}$$

and find that now

$$(-\omega^2 + 2i\gamma\omega + \omega_0^2)Ce^{i\omega t} = \frac{F_0}{m}e^{i\omega t}. \tag{3.80}$$

so

$$C = \frac{F_0/m}{\omega_0^2 - \omega^2 + 2i\gamma\omega}. \tag{3.81}$$

Recall that any complex number can be written in polar form

$$C = Ae^{-i\phi} \tag{3.82}$$

where A and ϕ are real numbers given by

$$A = |C| = \sqrt{CC^*}, \tag{3.83}$$

$$\phi = -\arctan\left(\frac{\text{Im } C}{\text{Re } C}\right). \tag{3.84}$$

(The minus sign is inserted in order to match the definition of ϕ used in the text book.)

Our solution can now be written as

$$x = \text{ Re } Ae^{i\omega t - \phi} = A\cos(\omega t - \phi) \tag{3.85}$$

The amplitude of the oscillation is

$$A = \sqrt{CC^*} = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}}, \tag{3.86}$$

and the phase shift is

$$\phi = \arctan\left(\frac{2\gamma\omega}{\omega_0^2 - \omega^2}\right). \tag{3.87}$$

The maximum amplitude occurs not at $\omega = \omega_0$, but at $\omega = \omega_r$, where

$$\omega_r^2 = \omega_0^2 - 2\gamma^2 = \omega_d^2 - \gamma^2, \tag{3.88}$$

which can be verified by taking the derivative of $A(\omega)$ and equating it to zero. Observe that if the the damping coefficient γ is too large, ω_r^2 will be negative, which means that there is no peak. This can be seen in the Figure 3.6.2 of the text book.

If the damping is not too strong, the amplitude at the peak will be

$$A(\omega_r) = \frac{F_0/m}{\sqrt{4\gamma^4 + 4\gamma^2(\omega_d^2 - \gamma^2)}} = \frac{F_0}{2m\gamma\omega_d} \tag{3.89}$$

and the ratio of the amplitude at the peak to the amplitude at zero frequency ($\omega = 0$) is

$$\frac{A(\omega_r)}{A(0)} = \frac{\omega_0^2}{2\gamma\omega_d} \simeq \frac{\omega_d}{2\gamma} = Q, \tag{3.90}$$

which is the Q of the system, defined earlier.

The sharpness of the peak can be described by the frequency difference $\Delta\omega$ between the frequencies at which which the energy (proportional to A^2) drops to half of its peak value. One finds that

$$\frac{\Delta\omega}{\omega_0} \simeq \frac{1}{Q}, \tag{3.91}$$

which shows that a system with a high Q value has a large, narrow, resonant peak.

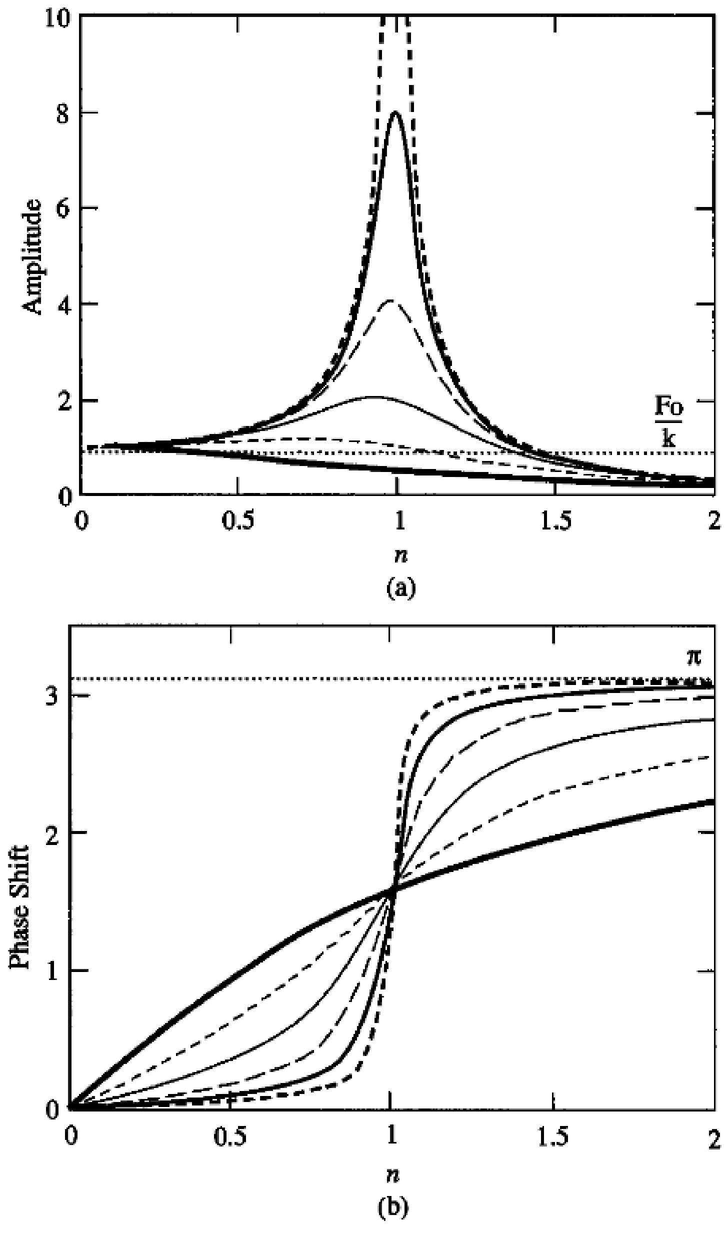

Figure 3.6.2 (a) Amplitude $A/(F_0/k)$ and (b) phase shift ϕ vs. driving frequency ($n = \omega/\omega_0$) for values of the damping constant γ given by $\gamma = 2^{-i}\,\omega_0$ ($i = 0, 1 \ldots 5$). Larger values of A and more abrupt phase shifts correspond to decreasing values of γ.

4 General motion of a particle in three dimensions

Let's now consider full three-dimensional motion. Recall that the position of a particle is described by a position vector $\boldsymbol{r}$. From this we can determine velocity, and acceleration by taking derivatives with respect to time. Newtons second law takes the form

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt}, \tag{4.1}$$

where $\boldsymbol{p} = m\boldsymbol{v} = m\dot{\boldsymbol{x}}$.

4.1 Conservative forces

Suppose that a moving particle is acted on by a force $\boldsymbol{F}(\boldsymbol{r})$ that is a function of position. The work done by this force on the particle as it makes an infinitesimal displacement $d\boldsymbol{r}$ is given by

$$dW = \boldsymbol{F} \cdot d\boldsymbol{r}. \tag{4.2}$$

Note the use of the dot product here. The work done depends only on the component of the force along the direction that the particle is moving.

The total work done by the force as the particle moves from point A to point B is obtained by dividing up the path into infinitesimal displacements and adding up all the contributions to the work,

$$W = \int_A^B \boldsymbol{F} \cdot d\boldsymbol{r}. \tag{4.3}$$

The integration is done along the path, or line, followed by the particle. It is called a ***line integral***. In general, a different path between the same points will result in a different amount of work being done.

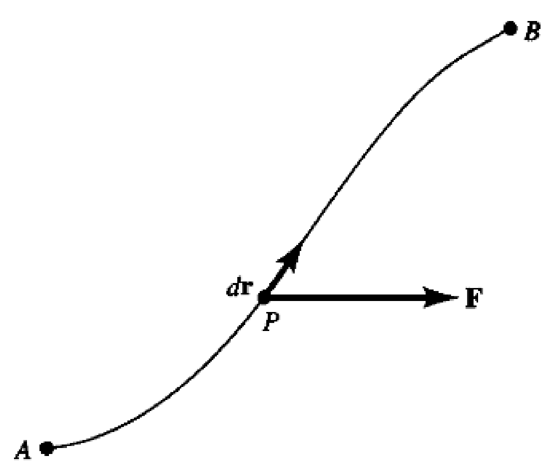

Figure 4.1.1 The work done by a force $\mathbf{F}$ is the line integral $\int_A^B \mathbf{F} \cdot d\mathbf{r}$.

However, if the force field is ***conservative***, the the work done will be *independent* of the path that the particle takes from A to B.

Equivalently, if a particle moves in a closed loop, returning to point A, in a conservative force field, the total work done by the force will be zero.

In one dimension, this will be the case if the force can be written as the derivative of a function $V(x)$ called the potential. Let's try to generalize this to three dimensions. We assume that there exists some scalar function $V(\boldsymbol{r}) = V(x, y, z)$ such that

$$F_x = -\frac{\partial V}{\partial x}, \quad F_y = -\frac{\partial V}{\partial y}, \quad F_z = -\frac{\partial V}{\partial z}. \tag{4.4}$$

Here we need to use partial derivatives ∂ since the function V depends on more than one variable. $\partial V/\partial x$ means "take the derivative of V with respect to x, while holding y and z constant".

Now we can evaluate the line integral. We write the dot product in terms of components,

$$\boldsymbol{F} \cdot d\boldsymbol{r} = F_x dx + F_y dy + F_z dz = -\left(\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz\right) = -dV \tag{4.5}$$

The last step follows because the expression in parenthesis is the total change in V resulting from the displacements dx, dy and dz.

If we now substitute this in the line integral we just get

$$W = -\int_A^B dV = V(A) - V(B). \tag{4.6}$$

we see that the work depends only on the end-points, and not on the path taken. Thus the force described by Eqn.(4.4) is conservative.

It follows that for a conservative force, the work done by the force on a particle whose path is any closed loop is zero.

4.2 Vector operators

4.2.1 Gradient

Eqn.(4.4) can be written more simply by introducing a vector differential operator $\boldsymbol{\nabla}$ which, when acting on a scalar function, has Cartesian components

$$\boldsymbol{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right). \tag{4.7}$$

With this notation we can write

$$\boldsymbol{F} = -\boldsymbol{\nabla} V = \boldsymbol{\nabla}(-V) \tag{4.8}$$

and we say that $\boldsymbol{F}$ is the ***gradient*** of the function $-V$.

The symbol $\boldsymbol{\nabla}$ is called *del* or *nabla*. When acting on a function $f(\boldsymbol{r})$, it is usually just called *grad*. It produces a vector that points in the direction in which f increases the fastest, and whose length is equal to the rate of increase of f with distance in that direction.

When applied to $-V$, it therefore points in the direction in which V is *decreasing* the fastest. Thus the force pushes the particle towards smaller values of V.

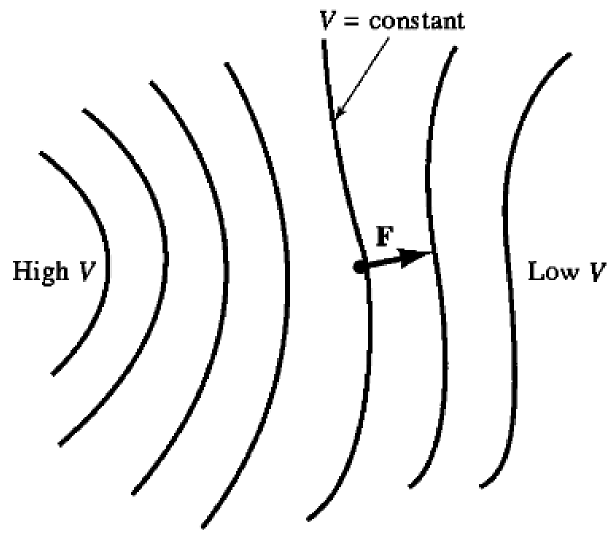

Figure 4.2.1 A force field represented by equipotential contour curves.

4.2.2 Curl

The line integral is related to another vector differential operator called the ***curl***. It is defined, in Cartesian coordinates in a manner analogous to the cross product. For any vector $\boldsymbol{A}$,

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \left(\frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y}\right)\boldsymbol{i} + \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\boldsymbol{j} + \left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\right)\boldsymbol{k} \tag{4.9}$$

We see that the curl of a vector field is another vector field.

Stokes theorem states that for any closed loop and *any* vector field $\boldsymbol{F}$,

$$\oint \boldsymbol{F} \cdot d\boldsymbol{r} = \int_S (\boldsymbol{\nabla} \times \boldsymbol{F}) \cdot d\boldsymbol{S}. \tag{4.10}$$

The right side of this equation is the integral over the surface S enclosed by the loop. The vector $d\boldsymbol{S}$ is a vector perpendicular to the surface whose length is equal to the infinitesimal element of area,

$$d\boldsymbol{S} = dydz\,\boldsymbol{i} + dzdx\,\boldsymbol{j} + dxdy\,\boldsymbol{k}. \tag{4.11}$$

(See Section 4.1 of Fowles and Cassiday, or any advanced calculus textbook, for a derivation of Stokes theorem. Fowles and Cassiday use the notation $\boldsymbol{n}da$ for $d\boldsymbol{S}$.)

Since the integral around any loop is zero if $\boldsymbol{F}$ is a conservative force, we see that an alternative condition for $\boldsymbol{F}$ to be conservative is that its curl must vanish everywhere.

In fact this condition is equivalent to the force being the gradient of a potential, because of the following vector identity, valid for any function f,

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} f = 0. \tag{4.12}$$

In other words, the curl of a gradient is always zero.

4.2.3 Divergence and Laplacian

The divergence operator produces a scalar field from a vector field $\boldsymbol{A}$. In Cartesian coordinates, it is defined by

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}. \tag{4.13}$$

Gauss's theorem relates the integral of a vector field over a *closed* surface to the integral of the divergence of the field over the enclosed volume,

$$\int_S \boldsymbol{A} \cdot d\boldsymbol{S} = \int_V \boldsymbol{\nabla} \cdot \boldsymbol{A}\, dV. \tag{4.14}$$

The ***Laplacian*** operator ∇^2 is a second-order differential operator that produces a scalar field from a scalar field. It is defined as the divergence of the gradient,

$$\nabla^2 f = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} f. \tag{4.15}$$

4.2.4 Helmholtz decomposition theorem

Any smooth (twice differentiable) vector field can be written as the sum of irrotational (zero curl) and divergenceless (zero divergence) vector fields,

$$F = -\boldsymbol{\nabla}\Phi + \boldsymbol{\nabla} \times \boldsymbol{A} \tag{4.16}$$

where

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi}\int_V \frac{\boldsymbol{\nabla}' \cdot F(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} dV' - \frac{1}{4\pi}\int_S \frac{\boldsymbol{F}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \cdot d\boldsymbol{S}' \tag{4.17}$$

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{4\pi}\int_V \frac{\boldsymbol{\nabla}' \times F(\boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|} dV' + \frac{1}{4\pi}\int_S \frac{\boldsymbol{F}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \times d\boldsymbol{S}' \tag{4.18}$$

(4.19)

Here V is some volume enclosing the point $\boldsymbol{r}$ and S is a surface enclosing that volume. If the F goes to zero sufficiently quickly (faster than $1/r$) at infinity, the surface integral will vanish if the volume is taken to be infinite.

4.2.5 Vector operator identities

There are many identities involving these operators. They are ***linear*** operators, which means $\mathcal{L}(af + bg) = a\mathcal{L}f + b\mathcal{L}g$, where $\mathcal{L}$ stands for div, grad or curl, a, b are constants and f, g are scalar or vector fields. Some non-intuitive identities are

$$\boldsymbol{\nabla}(\boldsymbol{A} \cdot \boldsymbol{B}) = (\boldsymbol{A} \cdot \boldsymbol{\nabla})\boldsymbol{B} + (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{A} + \boldsymbol{A} \times (\boldsymbol{\nabla} \times \boldsymbol{B}) + \boldsymbol{B} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) \tag{4.20}$$

$$\boldsymbol{\nabla} \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) - \boldsymbol{A} \cdot (\boldsymbol{\nabla} \times \boldsymbol{B}) \tag{4.21}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A}(\boldsymbol{\nabla} \cdot \boldsymbol{B}) - \boldsymbol{B}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) + (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{A} - (\boldsymbol{A} \cdot \boldsymbol{\nabla})\boldsymbol{B} \tag{4.22}$$

Useful identities involving two operators are

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} f = 0, \tag{4.23}$$

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) = 0, \tag{4.24}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}. \tag{4.25}$$

The last symbol, $\nabla^2 \boldsymbol{A}$ is the ***vector Laplacian***. It is a vector whose components are the Laplacian operator acting on each component of $\boldsymbol{A}$.

4.2.6 Curvilinear coordinates

It is often useful to employ coordinate systems that matched to the symmetry of the problem. We have already seen cylindrical and spherical coordinates. Other systems have also been devised, such as parabolic coordinates. In ***orthogonal curvilinear coordinate*** systems, lines traced by varying individual coordinates, leaving the others constant, always cross at right angles. As a result, the unit vectors are orthogonal at every point in space.

For a general orthogonal coordinate system having coordinates (q_1, q_2, q_3) and unit vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$, the element of length (called the ***line element***) can be written as

$$d\boldsymbol{r} = (h_1 dq_1)\boldsymbol{e}_1 + (h_2 dq_2)\boldsymbol{e}_2 + (h_3 dq_3)\boldsymbol{e}_3. \tag{4.26}$$

where the h's, called scale factors, are functions of the coordinates.

The element of volume is

$$dV = h_1 h_2 h_3 \, dq_1 \, dq_2 \, dq_3. \tag{4.27}$$

The gradient, divergence, curl and Laplacian are given by

$$\boldsymbol{\nabla} f = \frac{\boldsymbol{e}_1}{h_1}\frac{\partial f}{\partial q_1} + \frac{\boldsymbol{e}_2}{h_2}\frac{\partial f}{\partial q_2} + \frac{\boldsymbol{e}_3}{h_3}\frac{\partial f}{\partial q_3}, \tag{4.28}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(h_2 h_3 A_1) + \frac{\partial}{\partial q_2}(h_3 h_1 A_2) + \frac{\partial}{\partial q_3}(h_1 h_2 A_3)\right], \tag{4.29}$$

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \boldsymbol{e}_1 & h_2 \boldsymbol{e}_2 & h_3 \boldsymbol{e}_3 \\ \partial/\partial q_1 & \partial/\partial q_2 & \partial/\partial q_3 \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}, \tag{4.30}$$

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial f}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial q_3}\right)\right]. \tag{4.31}$$

The scale factors have the following values:

$$\begin{array}{llll} \text{Cartesian coordinates:} & h_x = 1 & h_y = 1 & h_z = 1, \\ \text{Cylindrical coordinates:} & h_R = 1 & h_\phi = R & h_z = 1, \\ \text{Spherical coordinates:} & h_r = 1 & h_\theta = r & h_\phi = r\sin\theta. \end{array} \tag{4.32}$$

4.3 Separable forces

Some three-dimensional problems are quite simple because the force can be written in the form

$$\boldsymbol{F} = F_x(x)\,\boldsymbol{i} + F_y(y)\,\boldsymbol{j} + F_z(z)\,\boldsymbol{k}. \tag{4.33}$$

Note that the x component is a function only of x, etc. It is easy to verify that $\boldsymbol{\nabla} \times \boldsymbol{F} = 0$, so the force is conservative.

The equation of motion is $m\ddot{\boldsymbol{r}} = \boldsymbol{F}$, which can only be true if the corresponding components of the vectors on the left and right sides are equal. Thus

$$m\ddot{x} = F_x(x), \quad m\ddot{y} = F_y(y), \quad m\ddot{z} = F_z(z). \tag{4.34}$$

These are three *independent* differential equations, each describing motion in one dimension. They can be solved separately using the methods that we have already discussed.

4.4 Motion of a projectile in a uniform gravitational field

As an example, consider a projectile launched with initial velocity v_0 at an angle α from the horizontal direction. We wish to determine the path it follows, and the distance and velocity when it hits the ground. The problem is two-dimensional so we set up a Cartesian coordinate system (x, z), with z being the vertical height.

4.4.1 No air resistance

In this case, the only force is that of gravity, acting in the $-z$ direction.

The equation of motion $m\ddot{\boldsymbol{r}} = \boldsymbol{F}$ has two components

$$m\ddot{x} = 0, \tag{4.35}$$
$$m\ddot{z} = -mg \tag{4.36}$$

The solutions are easily found,

$$x = At + B, \tag{4.37}$$
$$z = -\frac{1}{2}gt^2 + Ct + D. \tag{4.38}$$

where A, B, C, D are constants of integration.

We take the position at time $t = 0$ to be $x = z = 0$, which requires $B = D = 0$. The initial components of the velocity are

$$v_x(0) = \dot{x}(0) = v_0 \cos\alpha, \quad v_z(0) = \dot{z}(0) = v_0 \sin\alpha. \tag{4.39}$$

so $A = v_0 \cos\alpha$ and $C = v_0 \sin\alpha$. The solution is therefore

$$x = v_0 t \cos\alpha, \tag{4.40}$$
$$z = -\frac{1}{2}gt^2 + v_0 t \sin\alpha. \tag{4.41}$$

If we eliminate t by substitution of the first equation into the second we get an equation giving the height in terms of distance,

$$z = -\frac{g}{2v_0^2 \cos^2 \alpha} x^2 + x \tan \alpha \tag{4.42}$$

We see that the trajectory is a parabola. The distance travelled is found by setting $z = 0$ then solving for x. This gives

$$x(z = 0) = \frac{2}{g} v_0^2 \cos^2 \alpha \tan \alpha = \frac{v_0^2}{g} \sin(2\alpha). \tag{4.43}$$

For a given initial velocity, this is greatest when $\alpha = \pi/4 = 45°$,

$$x_{\text{max}} = v_0^2/g. \tag{4.44}$$

The maximum height can be found by equating the vertical velocity $\dot{z}$ to zero,

$$\dot{z} = -gt + v_0 \sin \alpha = 0, \tag{4.45}$$

which gives the time of maximum height

$$t(\dot{z} = 0) = \frac{v_0}{g} \sin \alpha. \tag{4.46}$$

The maximum height is then found by substituting this in the equation for z

$$z_{\text{max}} = -\frac{1}{2} g \left(\frac{v_0}{g} \sin \alpha \right)^2 + \frac{v_0^2}{g} \sin^2 \alpha = \frac{v_0^2}{2g} \sin^2 \alpha. \tag{4.47}$$

Note that these are all independent of the mass of the projectile. Also, we see that the maximum distance and height are proportional to the *square* of the initial velocity, and therefore proportional to the initial kinetic energy.

4.4.2 Including air drag

For a more realistic analysis, we should include air resistance. For any practical projectile, baseball, golf ball, artillery shell, etc, the drag will be dominated by the quadratic term.

Since the drag force is opposite to the velocity vector $\dot{\boldsymbol{r}}$, and has magnitude cv^2, we can write it in the form $F_{\text{drag}} = -cv\dot{\boldsymbol{r}}$. The equation of motion is now

$$m\ddot{\boldsymbol{r}} = -mg\boldsymbol{k} - cv\dot{\boldsymbol{r}}. \tag{4.48}$$

Unfortunately, the drag force is not separable as it involves all components ($v = \sqrt{\dot{x}^2 + \dot{y}^2}$). It also makes our differential equation non-linear.

Since we cannot solve this analytically, we try an approximation and set $v = v_0$. This makes the equation of motion linear and separable.

$$\ddot{x} = -\gamma \dot{x}, \tag{4.49}$$

$$\ddot{z} = -\gamma \dot{z} - g, \tag{4.50}$$

where

$$\gamma = \frac{cv_0}{m} \tag{4.51}$$

Integrating once gives the velocity components

$$\dot{x} = \dot{x}_0 e^{-\gamma t}, \tag{4.52}$$

$$\dot{z} = \dot{z}_0 e^{-\gamma t} - \frac{g}{\gamma}(1 - e^{\gamma t}) \tag{4.53}$$

A second integration gives the positions,

$$x = \frac{\dot{x}_0}{\gamma}(1 - e^{-\gamma t}), \tag{4.54}$$

$$z = \left(\frac{\dot{z}_0}{\gamma} + \frac{g}{\gamma^2}\right)(1 - e^{-\gamma t}) - \frac{g}{\gamma}t. \tag{4.55}$$

We can combine these to give a vector equation

$$\boldsymbol{r} = \left(\frac{\boldsymbol{v}_0}{\gamma} + \frac{g}{\gamma^2}\boldsymbol{k}\right)(1 - e^{-\gamma t}) - \frac{gt}{\gamma}\boldsymbol{k}. \tag{4.56}$$

The trajectory is no longer described by a parabola, but drops below it as the projectile looses energy. We can see from the equation for x that there is an upper limit to the distance that it can travel.

The distance at which it falls to earth can be found as before by setting $z = 0$. Solving the x equation for t we find

$$t = -\frac{1}{\gamma}\ln\left(1 - \frac{\gamma x}{\dot{x}_0}\right) \tag{4.57}$$

Substituting this in the z equation, we get

$$0 = \left(\dot{z}_0 + \frac{g}{\gamma}\right)\frac{x}{\dot{x}_0} + \frac{g}{\gamma^2}\ln\left(1 - \frac{\gamma x}{\dot{x}_0}\right). \tag{4.58}$$

This is a transcendental equation for x. We can get an approximate answer by using a series expansion

$$\ln(1 + u) = u - \frac{u^2}{2} + \frac{u^3}{3} + \cdots. \tag{4.59}$$

Using this we find

$$0 = \left(\dot{z}_0 + \frac{g}{\gamma}\right)\frac{x}{\dot{x}_0} - \frac{g}{\gamma^2}\left(\frac{\gamma x}{\dot{x}_0} + \frac{\gamma^2 x^2}{2\dot{x}_0^2} + \cdots\right). \tag{4.60}$$

which leads to

$$x_{\max} = \frac{2\dot{z}_0\dot{x}_0}{g} - \frac{8\dot{x}_0\dot{z}_0^2}{3g^2}\gamma + \cdots \tag{4.61}$$

In terms of v_0 and α, this is

$$x_{\max} = \frac{v_0^2\sin 2\alpha}{g} - \frac{4v_0^3\sin 2\alpha\sin\alpha}{3g^2}\gamma + \cdots \tag{4.62}$$

The first term is the range if there is no air resistance. The remaining terms correct for the effect of air resistance.

The results of numerical calculations for the trajectory of a baseball are shown in Figure 4.3.4 of the text book.

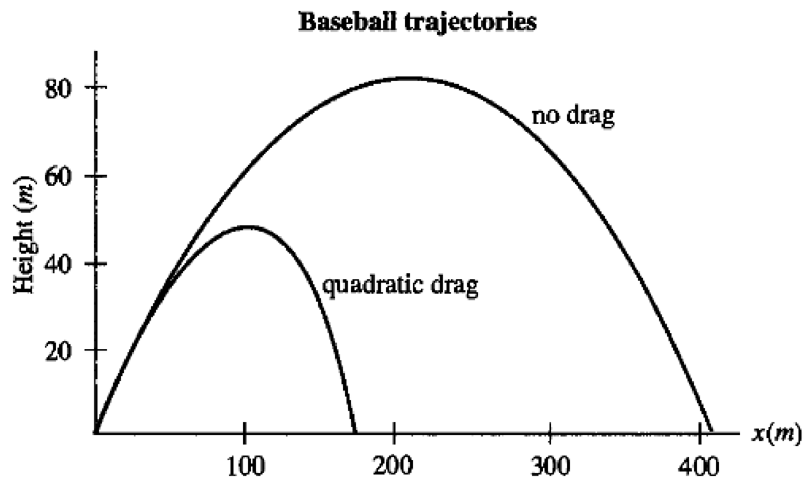

Figure 4.3.4 The calculated range of a baseball with quadratic air drag and without air drag. The range of the baseball is 172.2 m (565 ft) for an initial speed of 143.2 mph and elevation angle of 39 degrees.

4.5 Three dimensional harmonic oscillator

Suppose that a particle under a force that has the form $\boldsymbol{F} = -k\boldsymbol{r}$. The force is directed radially towards the origin and has a magnitude that is proportional to distance from the origin.

The equation of motion $m\ddot{\boldsymbol{r}} = -k\boldsymbol{r}$ separates into three equations

$$\begin{aligned} m\ddot{x} + kx &= 0, \\ m\ddot{y} + ky &= 0, \\ m\ddot{z} + kz &= 0. \end{aligned} \tag{4.63}$$

These are three independent harmonic oscillator equations, having the same frequency $\omega = \sqrt{k/m}$. The solutions can be written in the form

$$\begin{aligned} x &= A_x \sin\omega t + B_x \cos\omega t, \\ y &= A_y \sin\omega t + B_y \cos\omega t, \\ z &= A_z \sin\omega t + B_z \cos\omega t, \end{aligned} \tag{4.64}$$

where the six constants are determined by the initial conditions (three components of the initial position and three components of the initial velocity). These equations can be combined to form a single vector equation,

$$\boldsymbol{r} = \boldsymbol{A}\sin\omega_0 t + \boldsymbol{B}\cos\omega_0 t. \tag{4.65}$$

We see that $\boldsymbol{r}$ is a linear combination of the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$. It therefore lies in the plane that contains these two vectors. So, the motion is two-dimensional.

We can rotate the reference frame, using an orthogonal transformation $O \to O'$, so that the motion is confined to the $x' - y'$ plane. In the rotated frame, the constant vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ will have different components,

$$\boldsymbol{A} = A'_x \boldsymbol{i}' + A'_y \boldsymbol{j}' \tag{4.66}$$

$$\boldsymbol{B} = B'_x \boldsymbol{i}' + B'_y \boldsymbol{j}' \tag{4.67}$$

Now there are only two equations involving four constants, since $z' = 0$ for the orbit,

$$\begin{aligned} x' &= A'_x \sin\omega t + B'_x \cos\omega t, \\ y' &= A'_y \sin\omega t + B'_y \cos\omega t, \end{aligned} \tag{4.68}$$

We can solve this for $\sin\omega t$ and $\cos\omega t$,

$$\sin\omega t = \frac{1}{\Delta}(x'B'_y - y'B'_x), \quad \cos\omega t = \frac{1}{\Delta}(y'A'_x - x'A'_y). \tag{4.69}$$

where $\Delta = A'_x B'_y - A'_y B'_x$ is the determinant of the matrix of coefficients. Squaring these and adding, we get an equation for the path of the particle,

$$(A'^2_y + B'^2_y)x'^2 - 2(A'_x A'_y + B'_x B'_y)xy + (A'^2_x + B'^2_x)y^2 = \Delta^2 \tag{4.70}$$

This has the form of a general quadratic

$$ax'^2 + bx'y' + cy'^2 + dx' + ey' = f \tag{4.71}$$

This represents and ellipse, parabola or hyperbola depending on whether $b^2 - 4ac$ is negative, zero or positive, respectively. In this case

$$b^2 - 4ac = -(A'_x B'_y - A'_y B'_x)^2 \tag{4.72}$$

which is negative, and $d = e = 0$, so the path is an ellipse centred at the origin. Unless $b = 0$, the major axis of the ellipse is rotated with respect to the x' and y' axes.

4.6 Anisotropic harmonic oscillator

In the above, the constant k was independent of direction. This corresponds to a potential $V(\boldsymbol{r}) = kr^2/2$ that is spherically symmetric. More generally, the surfaces of constant potential could be ellipsoids. In a coordinate system aligned with the principal axes of the ellipsoid, the potential has the form

$$V = \frac{1}{2}k_1 x^2 + \frac{1}{2}k_2 y^2 + \frac{1}{2}k_3 z^2. \tag{4.73}$$

and the equation of motion $m\ddot{\boldsymbol{r}} = -\boldsymbol{\nabla} V$ separates to give the equations

$$\begin{aligned} m\ddot{x} + k_1 x &= 0, \\ m\ddot{y} + k_2 y &= 0, \\ m\ddot{z} + k_3 z &= 0. \end{aligned} \tag{4.74}$$

The solutions can be written in the form

$$\begin{aligned} x &= A_1 \sin(\omega_1 t + \phi_1), \\ y &= A_2 \sin(\omega_2 t + \phi_2), \\ z &= A_3 \sin(\omega_3 t + \phi_3), \end{aligned} \tag{4.75}$$

Again these are independent oscillators, but now they have different frequencies. The motion is confined to the interior of a box whose sides are $2A_1, 2A_2, 2A_3$. The path will close and repeat itself only if there exists integers n_1, n_2, n_3 such that

$$\frac{\omega_1}{n_1} = \frac{\omega_2}{n_2} = \frac{\omega_3}{n_3}. \tag{4.76}$$

The path is then a *Lissajous figure*. Otherwise, the path never repeats and the particle will eventually visit every point inside the box.

4.7 Constrained motion

Suppose that a body moves subject to a constraint. A simple pendulum is one example. Another would be a marble rolling inside a dish. We assume here that the constraint is frictionless and time independent.

The motion can be treated by adding a constraining force $\boldsymbol{R}$, corresponding to the force of reaction of the constraint. In the pendulum, it would be the tension in the arm or string. For the marble it is the force normal to the constraining surface.

The force of constraint is always perpendicular to the velocity of the particle and therefore does not add to or subtract from the energy of the body. If the other forces acting on the body are conservative, the total energy E will therefore be constant.

$$\frac{1}{2}mv^2 + V(\boldsymbol{r}) = E = constant. \tag{4.77}$$

This can be seen by writing the total force as $\boldsymbol{F} + \boldsymbol{R}$ and taking the dot prodoct of $\boldsymbol{v}$ with the equation of motion

$$m\boldsymbol{v} \cdot \dot{\boldsymbol{v}} + \boldsymbol{v} \cdot \boldsymbol{\nabla} V = \boldsymbol{v} \cdot \boldsymbol{R} = 0. \tag{4.78}$$

which can be written as

$$\frac{d}{dt}\left[\frac{1}{2}mv^2 + V\right] = 0. \tag{4.79}$$

5 Noninertial reference frames

In some situations it is convenient to use a non-inertial reference frame, such as a frame on the surface of the Earth, aligned with the horizontal and vertical directions. This frame rotates with a period of about 23 hrs 56 minutes (the *sidereal period* of the Earth).

Before examining rotating frames, we consider the simpler case of frames that are accelerating, but not rotating.

5.1 Non-rotating frames

Consider two reference frames. O is an inertial frame, and therefore not accelerating, and O' is an accelerating frame. Coordinates in the accelerating frame with be denoted by primes.

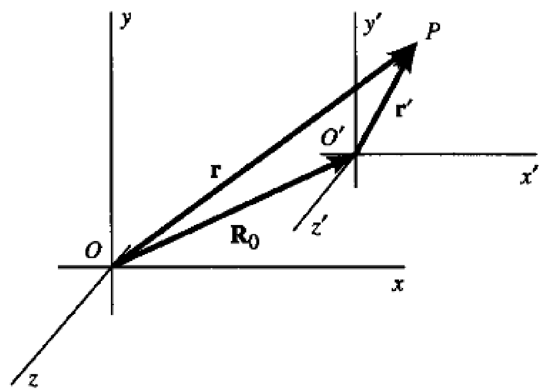

Figure 5.1.1 Relationship between the position vectors for two coordinate systems undergoing pure translation relative to each other.

In this case the frame O' is accelerating with respect to an inertial frame O, which may also be moving, and displaced, from O. Let the position, velocity and acceleration of the origin of the O' frame as seen from the O frame be R_0, V_0, and A_0. Then, the position, velocity and acceleration of a body as seen from the inertial frame is

$$\boldsymbol{r} = \boldsymbol{R_0} + \boldsymbol{r}', \tag{5.1}$$

$$\boldsymbol{v} = \boldsymbol{V_0} + \boldsymbol{v}', \tag{5.2}$$

$$\boldsymbol{a} = \boldsymbol{A_0} + \boldsymbol{a}', \tag{5.3}$$

Newton's second law is valid in the inertial system O, so for a particle of mass m,

$$\boldsymbol{F} = m\boldsymbol{a} = m\boldsymbol{A}_0 + m\boldsymbol{a}' \tag{5.4}$$

Now define $\boldsymbol{F}' = \boldsymbol{F} - m\boldsymbol{A_0}$. Then

$$\boldsymbol{F}' = m\boldsymbol{a}' \tag{5.5}$$

Thus, in the O' frame, the particle moves as if acted on by the force $\boldsymbol{F}'$. This force has two components. The first is the "true" force $\boldsymbol{F}$ and the second is a *ficticious* or *inertial* force $-m\boldsymbol{A}_0$ due to the acceleration of the reference frame.

Example (5.1.1 in the text book):

A block of wood rests on a rough horizontal table. If the table is accelerated in the horizontal direction, under what conditions will the block slip?

The block is prevented from slipping by friction, which exerts a force that has a maximum value of $\mu_s mg$, where μ_s is the coefficient of static friction and m is the mass of the block. The block will begin to slip if the magnitude of the inertial force mA_0 exceeds the maximum frictional force $\mu_s mg$. This gives the condition

$$A_0 > \mu_s g. \tag{5.6}$$

5.2 Rotating frames

Suppose now that O is an inertial frame and O' is a frame having the same origin but that O' is rotating with angular speed ω about an axis defined by the unit vector n. Define the *angular velocity* vector by

$$\boldsymbol{\omega} = \omega \boldsymbol{n} \tag{5.7}$$

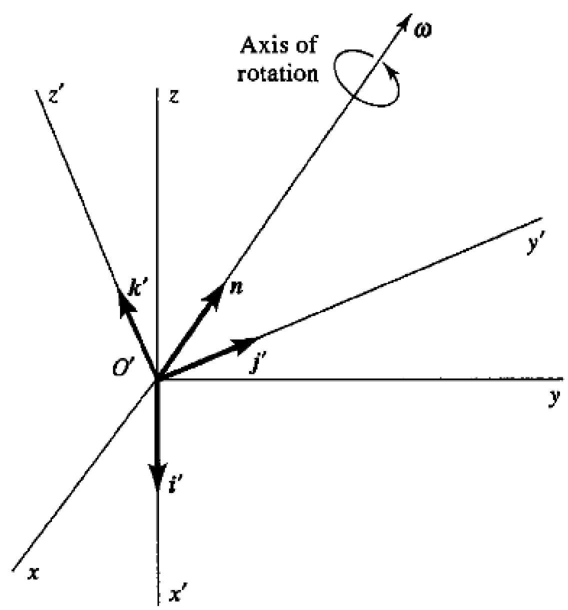

Figure 5.2.1 The angular velocity vector of a rotating coordinate system.

Any vector $\boldsymbol{Q}$ that appears fixed when viewed from the rotating frame, will be seen to be rotating with that frame when viewed from the inertial frame. Referring to Figure 5.1, we see that in time dt, the vector will change by an amount $dQ = Q \sin\theta d\phi$ in a direction perpendicular to both the vector $\boldsymbol{\omega}$ and the vector $\boldsymbol{Q}$. The angle that the frame rotates in this time is $d\phi = \omega dt$. Therefore we can write the rate of change of $\boldsymbol{Q}$ in terms of the cross product,

$$\frac{d\boldsymbol{Q}}{dt} = \boldsymbol{\omega} \times \boldsymbol{Q}. \tag{5.8}$$

Now suppose that the vector $\boldsymbol{Q}$ is moving even in the rotating frame. There will then be an additional change $d\boldsymbol{Q}_{\text{rot}}$, where the subscript indicates the change seen in the rotating frame. The

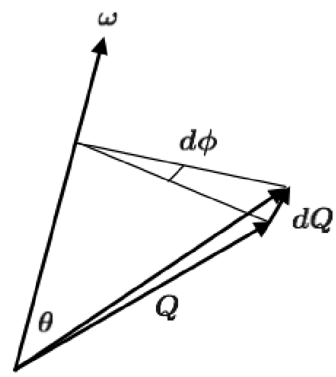

Figure 5.1: Change of a vector $\boldsymbol{Q}$ that is fixed in the rotating frame.

total change in $\boldsymbol{Q}$, as seen from the fixed (inertial) frame, is then $d\boldsymbol{Q}_{\text{rot}} + d\boldsymbol{Q}$. Thus the total rate of change seen in the fixed frame is

$$\left(\frac{d\boldsymbol{Q}}{dt}\right)_{\text{fixed}} = \left(\frac{d\boldsymbol{Q}}{dt}\right)_{\text{rot}} + \boldsymbol{\omega} \times \boldsymbol{Q}. \tag{5.9}$$

Lets now apply this result to relate the position, velocity and acceleration of a particle as seen from the rotating and fixed frames. Because the origins of the two frames coincide, $\boldsymbol{r} = \boldsymbol{r}'$.

To find the velocity, we must differentiate $\boldsymbol{r}$ with respect to time. Our previous result gives

$$\boldsymbol{v} = \left(\frac{d\boldsymbol{r}}{dt}\right)_{\text{fixed}} = \left(\frac{d\boldsymbol{r}}{dt}\right)_{\text{rot}} + \boldsymbol{\omega} \times \boldsymbol{r} = \boldsymbol{v}' + \boldsymbol{\omega} \times \boldsymbol{r}. \tag{5.10}$$

Since $\boldsymbol{r} = \boldsymbol{r}'$ we can write

$$\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{\omega} \times \boldsymbol{r}'. \tag{5.11}$$

To find the acceleration we differentiate the velocity,

$$\boldsymbol{a} = \left(\frac{d\boldsymbol{v}}{dt}\right)_{\text{fixed}} = \left(\frac{d\boldsymbol{v}}{dt}\right)_{\text{rot}} + \boldsymbol{\omega} \times \boldsymbol{v}. \tag{5.12}$$

Now substitute Equation (5.11) in Equation (5.12) to get

$$\begin{aligned} \boldsymbol{a} &= \left[\frac{d}{dt}(\boldsymbol{v}' + \boldsymbol{\omega} \times \boldsymbol{r}')\right]_{\text{rot}} + \boldsymbol{\omega} \times (\boldsymbol{v}' + \boldsymbol{\omega} \times \boldsymbol{r}'), \\ &= \boldsymbol{a}' + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}' + 2\boldsymbol{\omega} \times \boldsymbol{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}'). \end{aligned} \tag{5.13}$$

If we now include the possibility that the rotating frame is also moving and accelerating, the relations between vectors in the fixed and rotating frames become

$$\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{\omega} \times \boldsymbol{r}' + \boldsymbol{V}_0, \tag{5.14}$$

$$\boldsymbol{a} = \boldsymbol{a}' + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}' + 2\boldsymbol{\omega} \times \boldsymbol{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}') + \boldsymbol{A}_0. \tag{5.15}$$

The term $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}')$ is called the ***centripetal acceleration***. It points directly towards the axis of rotation.

The term $2\boldsymbol{\omega} \times \boldsymbol{v}'$ is called the ***Coriolis acceleration***. It occurs whenever an object moves in a rotating frame (unless $\boldsymbol{v}'$ is parallel to the axis of rotation).

The term $\dot{\boldsymbol{\omega}} \times \boldsymbol{r}'$ is called the ***transverse acceleration*** as it is always perpendicular to $\boldsymbol{r}'$. It occurs whenever the angular velocity (speed or direction) of the rotating frame changes.

5.3 Dynamics in non-inertial frames

Now that we have found the transformation rules, we can write down Newton's second law, which holds in the inertial frame,

$$\boldsymbol{F} = m\boldsymbol{a} = m\boldsymbol{a}' + m\dot{\boldsymbol{\omega}} \times \boldsymbol{r}' + 2m\boldsymbol{\omega} \times \boldsymbol{v}' + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}') + m\boldsymbol{A}_0. \tag{5.16}$$

If we now move everything except $m\boldsymbol{a}'$ to the left side of the equation and call this $\boldsymbol{F}'$, we obtain

$$\boldsymbol{F}' = m\boldsymbol{a}' \tag{5.17}$$

where

$$\boldsymbol{F}' = \boldsymbol{F} - m\dot{\boldsymbol{\omega}} \times \boldsymbol{r}' - 2m\boldsymbol{\omega} \times \boldsymbol{v}' - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}') - m\boldsymbol{A}_0. \tag{5.18}$$

For an observer in the rotating frame, the particle appears to move under the action of $\boldsymbol{F}$ plus a number of inertial forces. These are the ***transverse force*** $-m\dot{\boldsymbol{\omega}} \times \boldsymbol{r}'$, the ***Coriolis force*** $-2m\boldsymbol{\omega} \times \boldsymbol{v}'$, the ***centrifugal force*** $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}')$ and the inertial force $-m\boldsymbol{A}_0$.

5.4 Effects of the Earth's rotation

Let's now apply this theory to a specific example, a frame that is rotating with the Earth. We shall consider only the rotational motion of the Earth and ignore the small acceleration of the centre of the Earth as it orbits the Sun, and also the small acceleration of the centre of the Sun as it orbits the Galaxy.

The ***sideral period*** of the Earth is the rotation period with respect to an inertial coordinate system defined by distant stars. It is approximately 23 hours, 56 minutes and 4.0916 seconds (86,164.0916 s). In this time, the Earth rotates through an angle of 2π radians. This corresponds to an angular rate of

$$\omega_{\oplus} = 7.29211576 \times 10^{-5}\ \mathrm{s}^{-1}. \tag{5.19}$$

5.4.1 The plumb bob

A plumb bob is a weight hanging from a string, used to define the vertical direction. While it is generally perpendicular to the horizontal plane, it does not point directly towards the centre of the Earth, unless it is at a pole, or on the equator.

To see this, set up an inertial frame O centred on the Earth, and a rotating frame O' that has the bob (the weight) at its origin. In this frame, the acceleration of the bob is given by

$$\boldsymbol{a}' = \frac{1}{m}\boldsymbol{F}' = \boldsymbol{g} - \dot{\boldsymbol{\omega}} \times \boldsymbol{r}' - 2m\boldsymbol{\omega} \times \boldsymbol{v}' - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}') - \boldsymbol{A}_0. \tag{5.20}$$

where $\boldsymbol{g}$, the gravitational acceleration, points towards the centre of the Earth (although not exactly, see below). Since the bob is at rest at the origin, $\boldsymbol{v}' = \boldsymbol{r}' = 0$, so this reduces to

$$\boldsymbol{a}' = \boldsymbol{g} - \boldsymbol{A}_0, \tag{5.21}$$

This acceleration produces a force $m\boldsymbol{a}'$ that is opposed by the tension in the string. So, when the bob is at rest, the direction of the string is exactly aligned with the direction of $\boldsymbol{a}'$. Let us now find that direction.

The acceleration $\boldsymbol{A}_0$ of the origin of the O' frame is the centripetal acceleration due to the motion of the frame O' around the Earth's axis. It points directly towards this axis and has magnitude

$$A_0 = \omega_{\oplus}^2 R_{\oplus} \cos\lambda, \tag{5.22}$$

where λ is the latitude of the plumb bob. Both $\boldsymbol{A}_0$ and $\boldsymbol{g}$ lie in a plane that contains the Earth's axis (i.e. a plane of constant longitude). $\boldsymbol{a}'$ must lie in that plane too since it is a linear combination of these two vectors. To an observer in the O' frame, it appears that the plumb bob is affected by a slightly modified gravitational force, which we can denote as $m\boldsymbol{g}' = m\boldsymbol{a}'$.

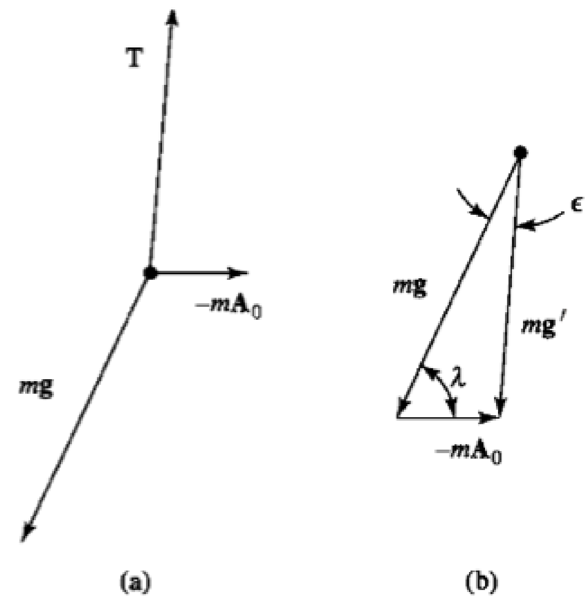

Figure 5.4.2 (a) Forces acting on a plumb bob at latitude λ. (b) Forces defining the weight of the plumb bob, $m\mathbf{g}'$.

Let ϵ be the angle between the vectors $\boldsymbol{g}'$ and $\boldsymbol{g}$. This angle is the deviation of the plumb bob caused by $\boldsymbol{A}_0$. To find it take the following dot products with Equation (5.21),

$$\boldsymbol{g}' \cdot \boldsymbol{g} = \boldsymbol{g} \cdot \boldsymbol{g} - \boldsymbol{A}_0 \cdot \boldsymbol{g}, \tag{5.23}$$

$$\boldsymbol{g}' \cdot \boldsymbol{a}' = \boldsymbol{g} \cdot \boldsymbol{g} - 2\boldsymbol{A}_0 \cdot \boldsymbol{g} + \boldsymbol{A}_0 \cdot \boldsymbol{A}_0. \tag{5.24}$$

From the definition of the dot product, $\boldsymbol{g} \cdot \boldsymbol{g} = g^2$, etc, and

$$\boldsymbol{g}' \cdot \boldsymbol{g} = g' g \cos\epsilon, \tag{5.25}$$

$$\boldsymbol{A}_0 \cdot \boldsymbol{g} = A_0 g \cos\lambda, \tag{5.26}$$

so we find that

$$\cos\epsilon = \frac{\boldsymbol{g}' \cdot \boldsymbol{g}}{g' g} = \frac{g - A_0 \cos\lambda'}{g} = \frac{g - A_0 \cos\lambda}{\sqrt{g^2 + A_0^2 - 2A_0 g \cos\lambda}}. \tag{5.27}$$

Since ϵ is small, $\sin\epsilon$ is more useful,

$$\sin\epsilon = \sqrt{1-\cos^2\epsilon} = \frac{A_0 \sin\lambda}{\sqrt{g^2 + A_0^2 - 2A_0 g\cos\lambda}}. \tag{5.28}$$

Now $A_0 \leqslant \omega_\oplus^2 R_\oplus \simeq 0.03 \ll g$ so a good approximation is

$$\epsilon \simeq \sin\epsilon \simeq \frac{A_0\sin\lambda}{g} = \frac{\omega_\oplus^2 R_\oplus}{g}\cos\lambda\sin\lambda = \frac{\omega_\oplus^2 R_\oplus}{2g}\sin 2\lambda \tag{5.29}$$

We see that the maximum deviation occurs for a latitude of $\pm 45°$ and has the value

$$\epsilon_{\max} = \frac{\omega_\oplus^2 R_\oplus}{2g} = 1.7\times 10^{-3}\text{ radian} \simeq 0.1°. \tag{5.30}$$

In fact $\boldsymbol{g}$ does not point exactly towards the centre of the Earth, since the Earth is slightly oblate, due to its rotation. Also, the direction of $\boldsymbol{g}$ depends on local terrain and density variations such as mountains, ore deposits, etc.

5.4.2 Ballistic motion

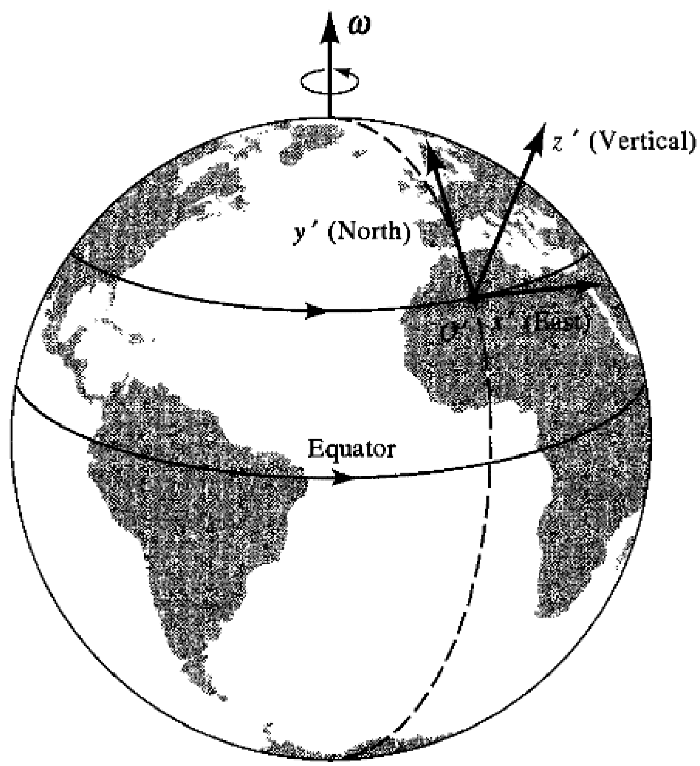

Figure 5.4.3 Coordinate axes for analyzing projectile motion.

Now consider the motion of a projectile, as seen from a reference frame rotating with the Earth. The equation of motion in this frame is

$$m\boldsymbol{a}' = \boldsymbol{F} + m\boldsymbol{g} - m\boldsymbol{A}_0 - 2m\boldsymbol{\omega}\times\boldsymbol{v}' - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{r}'). \tag{5.31}$$

Here $\boldsymbol{F}$ represents any force other than gravity. We saw in the previous example that $\boldsymbol{g} - \boldsymbol{A}_0 = \boldsymbol{g}'$, the apparent gravitational acceleration seen in the rotating frame. For this example, we shall assume that the projectile does not go so far that we need to consider variations of $\boldsymbol{g}'$ with latitude, and we take it to be constant. Also, we shall ignore air resistance, so $\boldsymbol{F} = 0$. Finally, for the Earth's rotation, $\omega = \omega_{\oplus} \ll 1$, so terms such as $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}')$ that have two or more factors of ω will be very small. We can integrate the equation of motion analytically if we keep only terms that are at most first order in ω. The equation of motion becomes

$$\boldsymbol{a}' = \boldsymbol{g}' - 2\boldsymbol{\omega} \times \boldsymbol{v}'. \tag{5.32}$$

Integrating this once, we get

$$\boldsymbol{v}' = \boldsymbol{g}'t - 2\boldsymbol{\omega} \times \boldsymbol{r}' + \boldsymbol{v}_0'. \tag{5.33}$$

We cannot integrate again due to the presence of $\boldsymbol{r}'$. But instead, we can substitute Equation (5.33) into (5.32) to get

$$\begin{aligned}\boldsymbol{a}' &= \boldsymbol{g}' - 2\boldsymbol{\omega} \times (\boldsymbol{g}'t - 2\boldsymbol{\omega} \times \boldsymbol{r}' + \boldsymbol{v}_0), \\ &= \boldsymbol{g}' - 2\boldsymbol{\omega} \times \boldsymbol{g}'t - 2\boldsymbol{\omega} \times \boldsymbol{v}_0'.\end{aligned} \tag{5.34}$$

In the last equation we have dropped the term $4\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}')$ as it is second-order in ω.

Now we can integrate this new equation to get the velocity and position vectors,

$$\boldsymbol{v}' = \boldsymbol{v}_0' + \boldsymbol{g}'t - 2\boldsymbol{\omega} \times \boldsymbol{v}_0 t - \boldsymbol{\omega} \times \boldsymbol{g}'t^2, \tag{5.35}$$

$$\boldsymbol{r}' = \boldsymbol{r}_0' + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}'t^2 - \boldsymbol{\omega} \times \boldsymbol{v}_0 t^2 - \frac{1}{3}\boldsymbol{\omega} \times \boldsymbol{g}'t^3. \tag{5.36}$$

We recognize the first few terms as they are the same as we found for projectile motion in an inertial frame. But now we have additional terms involving $\boldsymbol{\omega}$.

The term

$$-\boldsymbol{\omega} \times \boldsymbol{v}_0 t^2 \tag{5.37}$$

represents a horizontal deflection that is perpendicular to the velocity. The projectile moves in a curved path being deflected towards the right in the northern hemisphere and to the left in the southern hemisphere. Note that the amount of the deflection or bending of the trajectory is independent of the direction of motion. This is a result of the Coriolis force. It is the reason that air converging towards the centre of a low-pressure system or hurricane rotates about the centre in a counter-clockwise direction (clockwise in the southern hemisphere).

The term

$$-\frac{1}{3}\boldsymbol{\omega} \times \boldsymbol{g}'t^3 \tag{5.38}$$

also represents a horizontal deflection, but this is independent of the velocity. It is a deflection to the East (in both hemispheres) and increases with decreasing latitude, reaching a maximum at the equator.

5.4.3 The Foucault pendulum

A *spherical pendulum* is a pendulum that is free to move in two dimensions, such as a mass suspended from an overhead point by a thin wire. In an inertial frame, the equation of motion of the mass is

$$m\boldsymbol{a} = m\boldsymbol{g} + \boldsymbol{S} \tag{5.39}$$

where $\boldsymbol{S}$ is the tension in the wire. This tension has a vertical component $S_z = mg$, which opposes the gravitational force, and a horizontal component $-g\boldsymbol{R}/l$ that pushes the pendulum towards the vertical position. Since the pendulum is constrained by the wire, for small amplitudes in moves essentially in the horizontal plane and we can ignore the vertical components of the forces. The equation of motion is then that of a two-dimensional harmonic oscillator.

$$m\ddot{\boldsymbol{R}} = -\frac{g}{l}\boldsymbol{R} \tag{5.40}$$

The general solution is elliptical motion, but if the mass is pulled sideways and released with no initial velocity, it will swing back and forth radially, returning exactly to its starting position (in the absence of friction) with a period $T = 2\pi\sqrt{l/g}$.

In a frame O' fixed to the earth, the equation of motion of the mass must be modified to include the Coriolis force

$$m\boldsymbol{a}' = m\boldsymbol{g}' + \boldsymbol{S} - 2m\boldsymbol{\omega} \times \boldsymbol{v}' \tag{5.41}$$

(we have omitted the small term $m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}'))$. Here $\boldsymbol{\omega}$ is the angular velocity of the Earth, which points along the Earth's rotation axis, in the direction of the north pole.

As before, we need only consider the horizontal components of the forces. To do this we first resolve the angular velocity into the sum of vertical (z') and horizontal (x', y') components. Choosing the y' axis in the north direction, for latitude λ,

$$\boldsymbol{\omega} = \omega \sin\lambda\, \boldsymbol{k}' + \omega \cos\lambda\, \boldsymbol{j}' \tag{5.42}$$

Now, for small amplitudes, the velocity $\boldsymbol{v}$ is in the horizontal plane, as is the unit vector $\boldsymbol{j}'$, so the horizontal component of ω (the second term in the above equation) will produce a force that acts vertically. That force will be opposed by the tension in the wire and will not affect the motion of the pendulum. Keeping only the horizontal forces, the equation of motion becomes

$$m\ddot{\boldsymbol{R}}' = -\frac{g}{l}\boldsymbol{R}' - 2m\boldsymbol{\omega}' \times \boldsymbol{v}' \tag{5.43}$$

where we have defined $\boldsymbol{\omega}' = \omega \sin\lambda \boldsymbol{k}'$, the component of the Earth's angular velocity along the direction of the local vertical.

We can eliminate the second term by transforming to a new frame O'' that is rotating about the vertical axis with angular velocity $-\boldsymbol{\omega}'$ with respect to the frame O'. In this new frame, the Coriolis force disappears. The equation of motion becomes

$$m\ddot{\boldsymbol{R}}'' = -\frac{g}{l}\boldsymbol{R}'', \tag{5.44}$$

which is just the previous equation for a spherical pendulum. In this frame the pendulum swings back and forth returning to its original position.

But this frame is rotating with respect to the Earth, at a rate $-\omega \sin\lambda$. So in the O' frames, the the direction in which the pendulum swings slowly rotates. In other words, the pendulum *precesses* with a period

$$T = \frac{2\pi}{\omega_{\oplus} \sin\lambda}. \tag{5.45}$$

At the Earth's pole, the period is about 23 hours 56 minumtes (the Earth's sidereal period). The period gets longer as you approach the equator, becoming infinite there (no precession at all).

6 Gravitation and central forces

Having perfected Newtonian dynamics, Newton turned his attention to gravity. He formulated his Law of Universal Gravitation around 1665, during the plague, but did not publish it for more than 20 years. Newton was influenced by the laws of planetary motion established by Kepler in 1609 and 1618. Kepler's laws were empirical formulae based on extensive observations obtained by Tycho Brahe. in the preceeding decades.

Newton's insight was to realize that as the Moon circles the Earth, it is accellerating towards the Eaarth, just as if it were falling. Newton wondered if the Moon's acceleration could be caused by the same "gravitational" force that causes local objects, such as apples, to fall. This led him to the inverse square law, which he then found accurately explained all of Kepler's laws.

6.1 Newton's law of Universal Gravitation

Newton postulated that all bodies attract one another in proportion to the product of their masses and inversely proportional to the square of their separation. The force on one body acts directly towards the other body. Thus the force acting on a mass m_i due to a mass m_j is

$$\boldsymbol{F}_{ij} = G\frac{m_i m_j}{r_{ij}^3}\boldsymbol{r}_{ij}, \tag{6.1}$$

where $\boldsymbol{r}_{ij} = \boldsymbol{r}_j - \boldsymbol{r_i}$ is a vector extending from m_i to m_j and r_{ij} is its magnitude (the separation between the two masses). The proportionality constant G, called ***Newton's gravitational constant***, has the value $G = 6.6726 \times 10^{-11}$ N m^2 kg^{-2}.

Newton had no idea of why such a force existed, or how it was transmitted, and declined to speculate on this.is law of gravity was criticized on the basis of requiring some mysterious "action at a distance". Nevertheless, it correctly describes gravity (within certain limits).

A puzzling feature of Newton's law of gravity is that the force is exactly proportional to the mass that appears in his second law, the ***inertial mass***. Why inertia should be connected to gravity in this way is unknown, but it has been tested to very high accuracy by experiments such as that of Eötvos. The equality of gravitational and interial mass is called the ***principle of equivalence*** and it led Einstein to his geometrical theory of gravity, ***General Relativity***, which is more accurate that Newton's theory, but equivalent to it when the gravitational force is weak.

6.2 Newton's theorems

Newton also proved two useful theorems:

1. A spherically-symmetric mass affects external objects gravitationally as if all of its mass were concentrated at its centre.
2. A uniform shell of constant density exerts no net gravitational force on any object within it.

The proofs of these theorems rely on the inverse square dependence of the gravitational force. The first theorem is proved in the text book.

6.3 Kepler's laws

Kepler's three laws of planetary motion are

1. The orbit of a planet is an ellipse, with the Sun at a focus.
2. A line connecting the planet to the Sun sweeps out equal area in equal time.
3. The square of the orbital period is proportional to the cube of the semi-major axis of the ellipse.

Recall that an ellipse is the locus of points for which the sum of the distances from two points called ***foci*** is a constant. In a Cartesian coordinate system centred on the ellipse, its equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{6.2}$$

where a is the semi-major axis and $b \leqslant a$ is the semi-minor axis. The two focii are located at $x = \pm\epsilon a$, where

$$\epsilon = \sqrt{1 - b^2/a^2} \tag{6.3}$$

is called the eccentricity of the ellipse.

Another useful expression is the equation of an ellipse in polar coordinates, centred on one of the foci,

$$r = \frac{a(1-\epsilon^2)}{1+\epsilon\cos\theta}. \tag{6.4}$$

Here r is the distance from the focus to a point on the ellipse and θ is the angle of that point, measured from the point on the ellipse that is closest to the focus, called the ***pericentre*** or, for motion about the Sun, the ***perihelion***.

Since $-1 \leqslant \cos\theta \leqslant 1$ and $1 - \epsilon^2 = (1-\epsilon)(1+\epsilon)$, we that the distance from the focus to the pericentre is

$$r_p = a(1-\epsilon). \tag{6.5}$$

Similarly, the distance from the focus to the most distant point, the apocentre (or aphelion) is

$$r_a = a(1+\epsilon). \tag{6.6}$$

6.4 Kepler's second law, central forces and angular momentum

The ***angular momentum*** $\boldsymbol{L}$ of an object of mass m with respect to a point in an inertial frame, which we can take to be the origin, is defined by

$$\boldsymbol{L} = m\boldsymbol{r} \times \boldsymbol{v} \tag{6.7}$$

where $\boldsymbol{r}$ is the position vector and $\boldsymbol{v}$ is the velocity of the object. Since the (linear) momentum of the object is defined by $\boldsymbol{p} = m\boldsymbol{v}$, we can write this as

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} \tag{6.8}$$

Taking the time derivative,

$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{v} \times \boldsymbol{p} + \boldsymbol{r} \times \frac{d\boldsymbol{p}}{dt} = \boldsymbol{r} \times \boldsymbol{F}. \tag{6.9}$$

The first term is zero because $\boldsymbol{v}$ and $\boldsymbol{p}$ are parallel vectors and in the second term we have used Newton's second law $\boldsymbol{F} = d\boldsymbol{p}/dt$.

The second term $\boldsymbol{r} \times \boldsymbol{F}$ is the ***moment*** or ***torque*** acting on the particle with respect to the origin. If a second mass is located at the origin, the gravitational force that it exerts on the object is parallel to $\boldsymbol{r}$ and so the torque is zero. This is true for any ***central force***, regardless of the radial dependence of the force.

We conclude that the angular momentum of a planet orbiting the Sun will be a conserved quantity. The angular momentum $\boldsymbol{L}$ will not change its magnitude nor its direction. Since both $\boldsymbol{r}$ and $\boldsymbol{v}$ are perpendicular to $\boldsymbol{L}$, it follows that ***the orbit is confined to the plane that is perpendicular to*** $\boldsymbol{L}$. So, the orbit is two-dimensional and we can use polar coordinates (r, θ) in the orbital plane to describe it.

Kepler's second law states that the position vector $\boldsymbol{r}$ sweeps out equal areas in equal time. This is equivalent to the statement that

$$\frac{dA}{dt} = constant. \tag{6.10}$$

where dA is the area swept out in time dt. Referring to figure 6.4.1, we see that

$$dA = \frac{1}{2}\left|\boldsymbol{r} \times d\boldsymbol{r}\right|, \tag{6.11}$$

so

$$\frac{dA}{dt} = \frac{1}{2}\left|\boldsymbol{r} \times \frac{d\boldsymbol{r}}{dt}\right| = \frac{L}{2m} = \frac{l}{2}. \tag{6.12}$$

So we see that Kepler's third law is equivalent to the law of conservation of angular momentum, which follows from Newton's law of gravity as a central force.

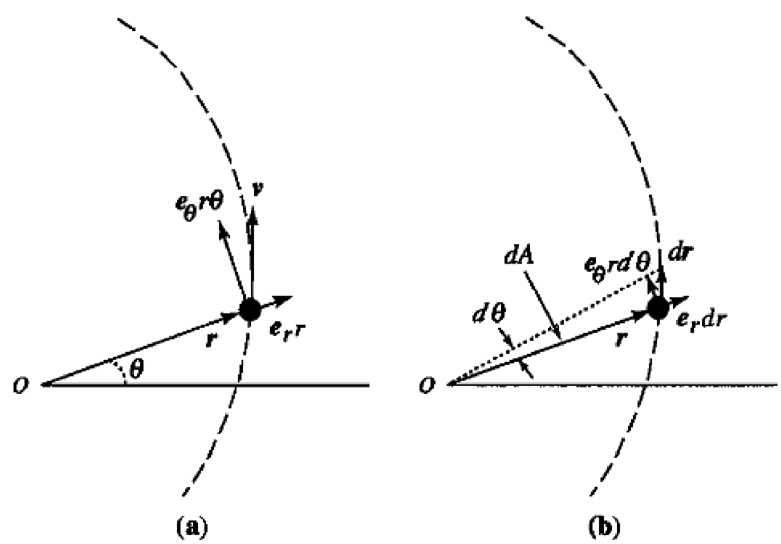

Figure 6.4.1 (a) Angular momentum $L = |\mathbf{r} \times m\mathbf{v}|$ of a particle moving in a central field. (b) Area $dA = \frac{1}{2}|\mathbf{r} \times d\mathbf{r}|$ swept out by the radius vector $\mathbf{r}$ of the particle as it moves in a central field.

6.5 Kepler's first law

We shall follow the development in the text book and first consider an arbitrary central force. This is then specialized to an inverse square law. We assume that the mass of the planet is negligible compared to that of the Sun.

Set up a polar coordinate system (r,θ) centred on the Sun. Since the mass of the planets is small, we assume that their gravity does not significantly perturb the motion of the Sun, so this frame is approximately inertial.

The equation of motion is

$$m\ddot{\boldsymbol{r}} = f(r)\boldsymbol{e}_r \tag{6.13}$$

where m is the mass of the planet and $f(r)$ is the magnitude of the force acting on the planet. The radial and angular components of this equation are (see Section 1.16)

$$m(\ddot{r} - r\dot{\theta}^2) = f(r), \tag{6.14}$$

$$m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0 \tag{6.15}$$

The second equation tells us that

$$\frac{d}{dt}(r^2\dot{\theta}) = 0 \tag{6.16}$$

so

$$r^2\dot{\theta} = l \tag{6.17}$$

where l is a constant. From the previous section we see that $l = L/m$, the angular momentum per unit mass. This result does not depend on the form of $f(r)$. We see, as we found in the previous section, that ***the angular momentum of an orbiting point mass is conserved for any central force***.

To find the shape of the orbit, we need an equation that gives r as a function of θ. Therefore we need to eliminate t from these equations. This is easily done using the last equation, which tells us that

$$\frac{d}{dt} = \frac{l}{r^2}\frac{d}{d\theta}. \tag{6.18}$$

The radial equation can therefore be written as

$$\frac{l}{r^2}\frac{d}{d\theta}\left(\frac{l}{r^2}\frac{dr}{d\theta}\right) - \frac{l^2}{r^3} = \frac{1}{m}f(r) \tag{6.19}$$

This can be simplified by means of the substitution

$$r = \frac{1}{u} \tag{6.20}$$

which gives

$$-l^2u^2\frac{d^2u}{d\theta^2} - l^2u^3 = \frac{1}{m}f(1/u) \tag{6.21}$$

so

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{ml^2u^2}f(1/u) \tag{6.22}$$

This is the general equation for the orbit of a mass attracted by a central force.

Let's now consider the special case

$$f(r) = -\frac{k}{r^2} \tag{6.23}$$

where $k = GMm$ is a constant. The equation of the orbit now becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{k}{ml^2}. \tag{6.24}$$

This is the equation of a simple harmonic oscillator, with an additive constant. The constant is just an constant "force" applied to the oscillator and we saw in Section 3.4 that this results in an offset to the equilibrium position. Thus the solution is

$$u = A\cos(\theta - \theta_0) + \frac{k}{ml^2} \tag{6.25}$$

where A and θ_0 are constants. The latter depends on the direction that we choose to measure theta from, so we are free to set $\theta_0 = 0$. Substituting $u = 1/r$, we get

$$r = \frac{1}{A\cos(\theta) + k/ml^2} = \frac{ml^2/k}{1 + (Aml^2/k)\cos\theta}. \tag{6.26}$$

Compare this to the equation of an ellipse in polar coordinates (Equation 6.4),

$$r = \frac{a(1-\epsilon^2)}{1 + \epsilon\cos\theta} \tag{6.27}$$

We see that $Aml^2/k = \epsilon$ and

$$a(1-\epsilon^2) = \frac{ml^2}{k} = \frac{l^2}{GM}. \tag{6.28}$$

so the planet's orbital angular momentum per unit mass l is related to the parameters of the orbit by

$$l^2 = GMa(1-\epsilon^2). \tag{6.29}$$

You might wonder what the orbit would look like if gravity did not follow an inverse square law. There is a discussion of this in Section 6.13 of the text book. In that case the orbit would not be an ellipse, in fact it would not generally be any closed curve. Rather, the orbit would eventually fill the entire orbital plane between an inner and outer radius.

If the force law is nearly, but not exactly, an inverse square, the orbit can be described by an ellipse in which the major axis slowly ***precesses*** (rotates about the Sun). In fact this is observed for the planet Mercury and is due in part to a departure from an inverse square law according to Einstein's theory of gravity. The correct prediction of the precession of Mercury's orbit was an important early confirmation of the theory of General Relativity.

6.6 Kepler's third law

We have already found that the rate at which a line connecting a planet to the Sun sweeps out area at a rate that is proportional to the orbital angular momentum,

$$\frac{dA}{dt} = \frac{l}{2}. \tag{6.30}$$

To find the period T of the orbit, we multiply by dt and integrate,

$$\int_0^A dA = \frac{l}{2}\int_0^T dt \tag{6.31}$$

where $A = \pi ab$ is the total area of the ellipse. Therefore,

$$T = \frac{2\pi ab}{l} = \frac{2\pi a^2\sqrt{1-\epsilon^2}}{l}. \tag{6.32}$$

Squaring this and substituting from Equation (6.28),

$$T^2 = \frac{4\pi^2 a^4(1-\epsilon^2)}{GMa(1-\epsilon^2)}. \tag{6.33}$$

Which gives us Kepler's third law,

$$T^2 = \frac{4\pi^2 a^3}{GM}. \tag{6.34}$$

(The textbook denotes the orbital period by τ. Most astronomers use the symbol P for period.)

6.7 Gravitational potential energy

Consider a small particle of mass m acted on by a central force

$$\boldsymbol{F} = f(r)\boldsymbol{e}_r, \tag{6.35}$$

This force is conservative (to see this, calculate the curl of the above equation), and can therefore be written in terms of the gradient of a potential energy function,

$$\boldsymbol{F} = -\boldsymbol{\nabla} V = -\frac{dV}{dr}\boldsymbol{e}_r \tag{6.36}$$

where

$$f(r) = -\frac{dV}{dr}. \tag{6.37}$$

Thus

$$V(r) = -\int_{r_{\text{ref}}}^{r} f(r)dr, \tag{6.38}$$

where r_{ref} is a reference value of r at which the potential is defined to be zero. (Note that we can add any constant to V without changing the force. This gives us freedom to choose a distance at which the potential is zero). Forces, such as gravity, that vary inversely as a power of the distance, go to zero as $r \to \infty$, so r_{ref} is usually taken to be infinity. Then,

$$V(r) = \int_r^{\infty} f(r)dr, \tag{6.39}$$

The energy required to move an object from r to infinity against the force is $-V(r)$.

For gravity,

$$f(r) = -\frac{GMm}{r^2} = -\frac{d}{dr}\left(-\frac{GMm}{r}\right) \tag{6.40}$$

so

$$V(r) = -\frac{GMm}{r}. \tag{6.41}$$

It is often more convenient to use potential energy per unit mass, which is independent of the mass of the particle. So we define the ***gravitational potential*** Φ by

$$\Phi(r) = \frac{1}{m}V(r). \tag{6.42}$$

The gravitational force acting on a particle of mass m is therefore

$$\boldsymbol{F} = -m\boldsymbol{\nabla}\Phi.$$

6.8 Orbital energy

Consider a small mass m orbiting a large mass M, where $(m \ll M)$. The total energy is

$$E = \frac{1}{2}mv^2 + V(r) \tag{6.43}$$

To find the energy, evaluate this expression when the object is at perihelion. $r = r_p$. At this point the radial component of the velocity is zero, so $v = l/r_p$. Thus

$$E = \frac{ml^2}{2r_p^2} - \frac{GMm}{r_p} = \frac{GMma(1-\epsilon^2)}{2r_p^2} - \frac{GMm}{r_p} = \frac{GMm}{r_p}\left[\frac{a(1-\epsilon^2)}{2r_p} - 1\right]. \tag{6.44}$$

Now substitute $r_p = a(1-\epsilon)$.

$$E = \frac{GMm}{a(1-\epsilon)}\left[\frac{1-\epsilon^2}{2(1-\epsilon)} - 1\right] = \frac{GMm}{2a(1-\epsilon)}\left(1+\epsilon-2\right), \tag{6.45}$$

which gives the total energy

$$E = -\frac{GMm}{2a}. \tag{6.46}$$

Recall that the geometry of the orbit is related to the angular momentum by Equation (6.28).

$$a(1-\epsilon^2) = \frac{l^2}{GM} = -\frac{ml^2}{2aE}. \tag{6.47}$$

so

$$\epsilon = \sqrt{1 + \frac{ml^2}{2a^2E}}. \tag{6.48}$$

We see from this that if $l = 0$, $\epsilon = 1$ which corresponds to a radial orbit. The object falls directly into the Sun. For nonzero l, if E is negative, then $\epsilon < 1$ so the orbit is elliptical (or circular) and the object is said to be ***bound*** to the Sun. If $E > 0$ then $\epsilon > 1$ and the orbit is a hyperbola. In this case the object is not bound to the Sun. It approaches from a great distance, passes by the Sun and then recedes to infinite distance. A final case is $E = 0$. Then, $a = \infty$ and $\epsilon = 1$. The orbit is a parabola.

6.9 Orbital velocity

We can rearrange the energy equation to find the velocity of the object at any point in its orbit,

$$v^2 = \frac{2E}{m} - \frac{2V}{m} = -\frac{GM}{a} + \frac{2GM}{r}, \tag{6.49}$$

so

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right). \tag{6.50}$$

For a circular orbit, $r = a$ and we see that the ***circular velocity*** is

$$v = \sqrt{\frac{GM}{r}} \tag{6.51}$$

6.10 Repulsive inverse-square forces - Rutherford scattering

We can use the results of the previous sections to study a completely different problem, that of the scattering of alpha particles by a thin foil of gold. The unexpected result of this experiment led to the discovery of the atomic nucleus.

In SI units, the electrostatic force between a charge Q and a charge q is given by Coulomb's law,

$$f(r) = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r^2}, \tag{6.52}$$

where $\varepsilon_0 = 8.854188 \times 10^{-12}$ F/m (= C^2/Nm^2) is the ***permittivity of free space***.

Suppose that the charge q has a small mass m, which is negligible compared to the large mass M of the charge Q. The motion of the charge q can then be found by exactly the same analysis that we used for gravity, provided that we set the force constant to

$$k = -\frac{Qq}{4\pi\varepsilon_0}. \tag{6.53}$$

We obtain the same equation for the orbit (6.26), but now the eccentricity is given by

$$a(1-\epsilon^2) = \frac{ml^2}{k} = -\frac{4\pi\varepsilon_0 ml^2}{Qq}, \tag{6.54}$$

so

$$\epsilon = \sqrt{1 + \frac{4\pi\varepsilon_0 ml^2}{aQq}}. \tag{6.55}$$

We see that if Q and q have the same sign, $\epsilon > 1$ and the path of the small particle is a hyperbola (see Figure 6.14.1).

In scattering problems, it is convenient to express the results in terms of the particle energy E and the impact parameter b. If the particle has initial velocity v_0 when far away, its total energy is the same as the initial kinetic energy,

$$E = \frac{1}{2}mv_0^2. \tag{6.56}$$

and the angular momentum is the same as the initial angular momentum

$$l = v_0 b. \tag{6.57}$$

Using our previous relations for the orbital parameters in terms of energy and angular momentum, we find

$$a = -\frac{k}{2E} = \frac{Qq}{8\pi\varepsilon_0 E}, \tag{6.58}$$

$$\epsilon^2 - 1 = -\frac{ml^2}{ka} = 2(4\pi\varepsilon_0)^2 \frac{mEl^2}{Q^2 q^2}. \tag{6.59}$$

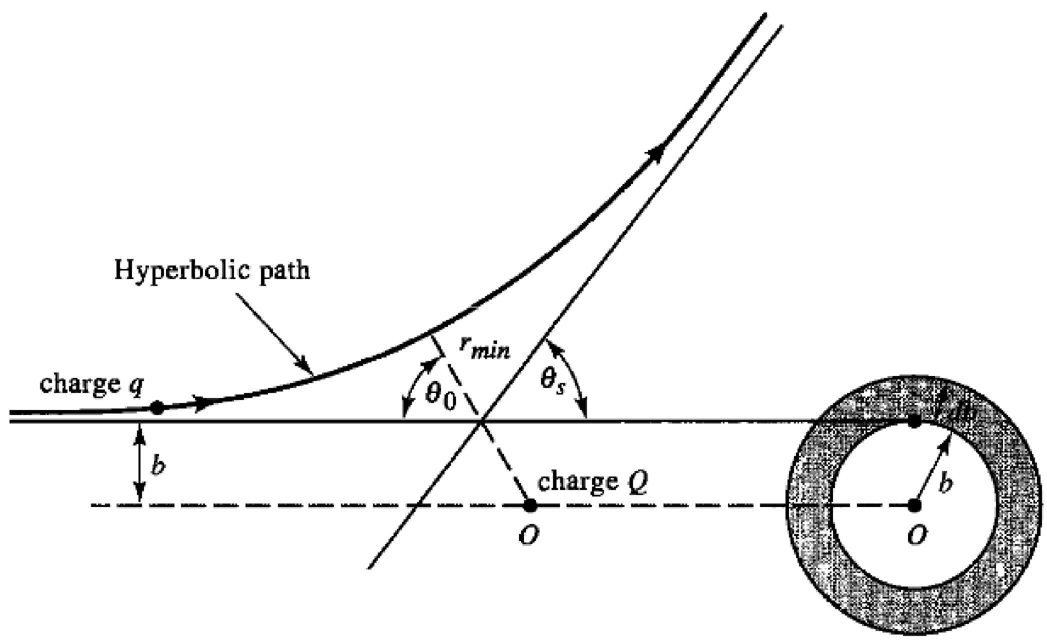

Figure 6.14.1 Hyperbolic path (orbit) of a charged particle moving in the inverse-square repulsive force field of another charged particle.

Let's find the scattering angle θ_s shown in the figure. The equation of the orbit is

$$r = \frac{a(1-\epsilon^2)}{1+\epsilon\cos(\theta+\pi)} = \frac{4\pi\varepsilon_0 ml^2/Qq}{-1+\epsilon\cos(\theta)}, \tag{6.60}$$

where the constant angle π is chosen so that the closest approach occurs when $\theta = 0$. We see that $r \to \infty$ when $\theta = \pm\theta_0$, where $\cos\theta_0 = 1/\epsilon$. The scattering angle is therefore

$$\theta_s = \pi - 2\theta_0. \tag{6.61}$$

From this it follows that

$$\cot(\theta_s/2) = \tan\theta_0 = \sqrt{\epsilon^2 - 1} = \frac{4\pi\varepsilon_0 l}{Qq}\sqrt{2Em} = \frac{4\pi\varepsilon_0 bmv_0^2}{Qq} = \frac{8\pi\varepsilon_0 bE}{Qq}. \tag{6.62}$$

For an individual scattering event, it is practically impossible to measure the impact parameter. However, in a typical experiment many particles are scattered from many nuclei so one can use a statistical approach.

Let N be the number of incident particles and n be the number of atoms per unit area in the gold foil. Then, the number of particles dN scattered with impact parameter between b and $b + db$ will be the same as the number of particles having an impact parameter that fall inside the shaded area shown in Figure 6.14.1. This is equal to this area multiplied by the number of atoms per unit area multiplied by the number of incident particles,

$$dN(b) = Nn(2\pi b db) \tag{6.63}$$

These particles will be scattered by an angle between θ_s and $\theta_s + d\theta_s$, which corresponds to a solid angle $d\Omega = 2\pi \sin\theta_s d\theta_s$. If we define the ***differential scattering cross section*** $\sigma(\theta_s)$ by

$$\sigma(\theta_s) = \frac{1}{Nn}\frac{dN}{d\Omega}, \tag{6.64}$$

then it follows that

$$dN = Nn\sigma(\theta_s)(2\pi \sin\theta_s d\theta_s) = Nn2\pi b db, \tag{6.65}$$

so

$$\sigma(\theta_s) = \frac{b}{\sin\theta_s}\left|\frac{db}{d\theta_s}\right|. \tag{6.66}$$

If we differentiate Equation (6.62) with respect to θ_s we get

$$\frac{1}{2\sin^2(\theta_s/2)} = \frac{8\pi\varepsilon_0 E}{Qq}\left|\frac{db}{d\theta_s}\right|. \tag{6.67}$$

We can now eliminate b by substituting this, and Equation (6.62) into Equation (6.66). The result is

$$\sigma(\theta_s) = \frac{Q^2 q^2 \cot(\theta_s/2)}{128\pi^2\varepsilon_0^2 E^2 \sin\theta_s \sin^2(\theta_s/2)}. \tag{6.68}$$

This can be simplified using the trigonometric identity

$$\sin\theta_s = 2\sin(\theta_s/2)\cos(\theta_s/2), \tag{6.69}$$

which gives

$$\sigma(\theta_s) = \frac{Q^2 q^2}{256\pi^2\varepsilon_0^2 E^2 \sin^4(\theta_s/2)}. \tag{6.70}$$

This famous result was obtained by Rutherford in 1911 and agrees very well with experimental measurements.

7 Multi-particle systems

7.1 Centre of mass and linear momentum

Consider a system of n particles having masses $m_1, \ldots, m_n$, positions $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_n$ and velocities $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$. The ***centre of mass*** of the system is defined by

$$\boldsymbol{r}_{\rm cm} = \frac{1}{m} \sum_{i=1}^{n} m_i \boldsymbol{r}_i, \tag{7.1}$$

where

$$m = \sum_{i=1}^{n} m_i \tag{7.2}$$

is the total mass. We see that the centre of mass is just the mean of the positions of the particles, weighted by the mass of each particle.

If we take the derivative of this with respect to time, we get

$$\boldsymbol{p} \equiv m\dot{\boldsymbol{r}}_{\rm cm} = \sum_i m_i \dot{\boldsymbol{r}}_i = \sum_i \boldsymbol{p}_i. \tag{7.3}$$

which is the ***total linear momentum*** of the system. For simplicity we will use the notation

$$\sum_i \equiv \sum_{i=1}^{n}. \tag{7.4}$$

Now suppose that the particles are acted upon by external forces $\boldsymbol{F}_i$, representing the external force on particle i, and internal forces $\boldsymbol{F}_{ij}$, representing the force acting on particle i due to particle j.

Newton's second law tells us that

$$\dot{\boldsymbol{p}}_i = \boldsymbol{F}_i + \sum_j \boldsymbol{F}_{ij}. \tag{7.5}$$

Therefore,

$$\dot{\boldsymbol{p}} = \sum_i \boldsymbol{F}_i + \sum_{i,j} \boldsymbol{F}_{ij}. \tag{7.6}$$

In the second term, for every term, such as $\boldsymbol{F}_{23}$ there is a corresponding term $\boldsymbol{F}_{32}$. But by Newton's third law these forces are equal and opposite, $\boldsymbol{F}_{ij} = -\boldsymbol{F}_{ji}$. Therefore the last term sums to zero,

$$\dot{\boldsymbol{p}} = \sum_i \boldsymbol{F}_i. \tag{7.7}$$

This tells us that the rate of change of the total linear momentum equals the sum of the external forces. *If there are no external forces (or if they add to zero), $\dot{\boldsymbol{p}} = 0$ and the momentum is conserved.*

7.2 Angular momentum

The total angular momentum of the system is defined by

$$\boldsymbol{L} = \sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i. \tag{7.8}$$

From this definition, it follows that the rate of change of the angular momentum is given by

$$\frac{d\boldsymbol{L}}{dt} = \sum_i \dot{\boldsymbol{r}}_i \times \boldsymbol{p}_i + \sum_i \boldsymbol{r}_i \times \dot{\boldsymbol{p}}_i = \sum_i \boldsymbol{r}_i \times \dot{\boldsymbol{p}}_i. \tag{7.9}$$

The first term vanishes because $\dot{\boldsymbol{r}}_i = \boldsymbol{v}_i$ is parallel to $\boldsymbol{p}_i = m_i \boldsymbol{v}_i$.

Using Newton's second law,

$$\frac{d\boldsymbol{L}}{dt} = \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i + \sum_{i,j} \boldsymbol{r}_i \times \boldsymbol{F}_{ij}. \tag{7.10}$$

The second term vanishes if the internal forces are central. To see this write

$$\sum_{i,j} \boldsymbol{r}_i \times \boldsymbol{F}_{ij} = \frac{1}{2}\left(\sum_{i,j} \boldsymbol{r}_i \times \boldsymbol{F}_{ij} + \sum_{j,i} \boldsymbol{r}_j \times \boldsymbol{F}_{ji}\right) = \frac{1}{2}\sum_{i,j}(\boldsymbol{r}_i - \boldsymbol{r}_j) \times \boldsymbol{F}_{ij} = \frac{1}{2}\sum_{i,j} \boldsymbol{r}_{ij} \times \boldsymbol{F}_{ij} = 0, \tag{7.11}$$

where $\boldsymbol{r}_{ij} = \boldsymbol{r}_i - \boldsymbol{r}_j$ is the vector extending from particle j to particle i, which is parallel to F_{ij} for a central force.

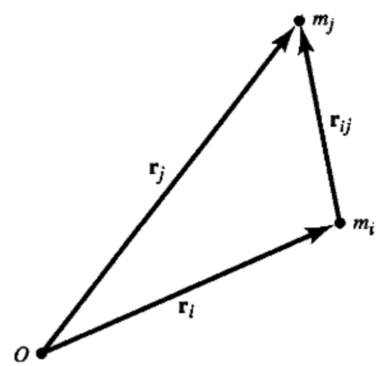

Figure 7.2.1 Definition of the vector $\mathbf{r}_{ij}$.

In the first term, $\boldsymbol{r}_i \times \boldsymbol{F}_i$ is the moment of the external force on particle i. The sum over i gives the total moment $\boldsymbol{N}$ of the external forces. Thus,

$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{N}. \tag{7.12}$$

If the total moment of the external forces is zero (i.e. there is no net torque), then $d\boldsymbol{L}/dt = 0$ and the total angular momentum of the system is conserved.

This derivation will fail if the internal forces are not central. For example the particles could be charged and therefore subject to electromagnetic forces. However, if the angular momentum present in the electromagnetic field generated by the particles is also included, one finds that the total angular momentum is conserved.

7.3 Centre of mass frame

If there is no net external force, the centre of mass is fixed, or moves with a constant velocity. We can then define an inertial frame called the ***centre of mass frame***, in which the position of the centre of mass is fixed, and we can choose it as the origin of our coordinate system.

Following the text book, we denote postions, velocities, momentum in the centre of mass frame with a bar, for example, $\bar{\boldsymbol{r}}_i$.

In the centre of mass frame, $\boldsymbol{r}_{\rm cm} = 0$, so it follows that

$$\sum_i m_i \bar{\boldsymbol{r}}_i = 0. \tag{7.13}$$

The derivative of this with respect to time gives

$$\sum_i m_i \bar{\boldsymbol{v}}_i = \sum_i \bar{\boldsymbol{p}}_i = 0. \tag{7.14}$$

We can write the position in any frame as the sum of the position of the centre of mass in that frame and the position of the particle in the centre of mass frame,

$$\boldsymbol{r}_i = \boldsymbol{r}_{\rm cm} + \bar{\boldsymbol{r}}_i. \tag{7.15}$$

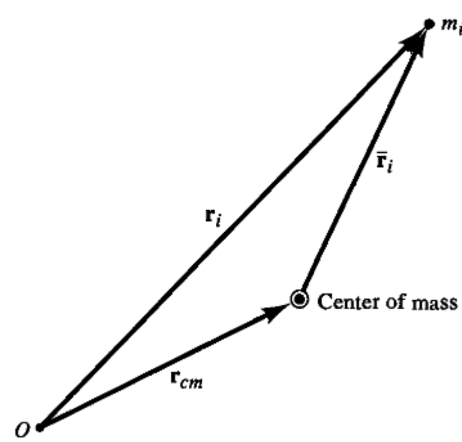

Figure 7.2.2 Definition of the vector $\bar{\mathbf{r}}_i$.

The derivative gives a corresponding relation for velocities,

$$\boldsymbol{v}_i = \boldsymbol{v}_{\rm cm} + \bar{\boldsymbol{v}}_i. \tag{7.16}$$

With a little algebra (see the text book), one finds that

$$\boldsymbol{L} = \boldsymbol{r}_{\rm cm} \times \boldsymbol{p}_{\rm cm} + \sum_i \bar{\boldsymbol{r}}_i \times \bar{\boldsymbol{p}}_i \tag{7.17}$$

This shows that the total angular momentum is the sum of an *orbital* part, due to the motion of the centre of mass, and a *spin* part involving rotation about the centre of mass.

7.4 Kinetic energy

The total kinetic energy is the sum of the kinetic energy of the individual particles,

$$T = \frac{1}{2}\sum_i m_i \boldsymbol{v}_i \cdot \boldsymbol{v}_i \tag{7.18}$$

Since $\boldsymbol{v}_i = \boldsymbol{v}_{\rm cm} + \bar{\boldsymbol{v}}_i$, this can be written as

$$\begin{aligned} T &= \frac{1}{2}\sum_i m_i(\boldsymbol{v}_{\rm cm} + \bar{\boldsymbol{v}}_i)\cdot(\boldsymbol{v}_{\rm cm} + \bar{\boldsymbol{v}}_i) \\ T &= \frac{1}{2}\sum_i m_i \boldsymbol{v}_{\rm cm}\cdot\boldsymbol{v}_{\rm cm} + \boldsymbol{v}_{\rm cm}\cdot\sum_i m_i\bar{\boldsymbol{v}}_i + \frac{1}{2}\sum_i m_i \bar{\boldsymbol{v}}_i\cdot\bar{\boldsymbol{v}}_i \\ &= \frac{1}{2}mv_{\rm cm}^2 + \frac{1}{2}\sum_i m_i \bar{v}_i^2. \end{aligned} \tag{7.19}$$

We see that the kinetic energy is the sum of the energy of motion of the centre of mass and the internal kinetic energy of the system.

7.5 Two-body motion

In the previous section on planetary motion, we assumed that the mass of the planet is negligible compared to that of the Sun. Lets now consider the case of two comparable masses.

Let the masses be m_1 and m_2. It is simplest to work in the centre-of-mass frame, with the origin at the centre of mass. The position vectors $\bar{\boldsymbol{r}}_1$ and $\bar{\boldsymbol{r}}_2$ then satisfy the condition

$$m_1\bar{\boldsymbol{r}}_1 + m_2\bar{\boldsymbol{r}}_2 = 0 \tag{7.20}$$

Therefore,

$$\bar{\boldsymbol{r}}_1 = -\frac{m_2}{m_1}\bar{\boldsymbol{r}}_2. \tag{7.21}$$

Both masses orbit about their common centre of mass, staying exactly opposite to each other. The ratio of their distances from the centre of mass is constant. The gravitational force acting on each mass is directed towards the other mass, and therefore points directly towards the centre of mass.

Let $\boldsymbol{R} = \bar{\boldsymbol{r}}_1 - \bar{\boldsymbol{r}}_2$ be the vector extending from m_1 to m_2. The equation of motion of m_1 is

$$m_1\ddot{\bar{\boldsymbol{r}}}_1 = -\frac{Gm_1m_2}{R^2}\boldsymbol{e}_R. \tag{7.22}$$

where $\boldsymbol{e}_R = \boldsymbol{R}/R$ is a unit vector in the $\boldsymbol{R}$ direction.

From the centre-of-mass relation,

$$m_1\bar{\boldsymbol{r}}_1 = -m_2(\bar{\boldsymbol{r}}_1 - \boldsymbol{R}) = 0 \tag{7.23}$$

so

$$\bar{\boldsymbol{r}}_1 = \frac{m_2}{m_1 + m_2}\boldsymbol{R}. \tag{7.24}$$

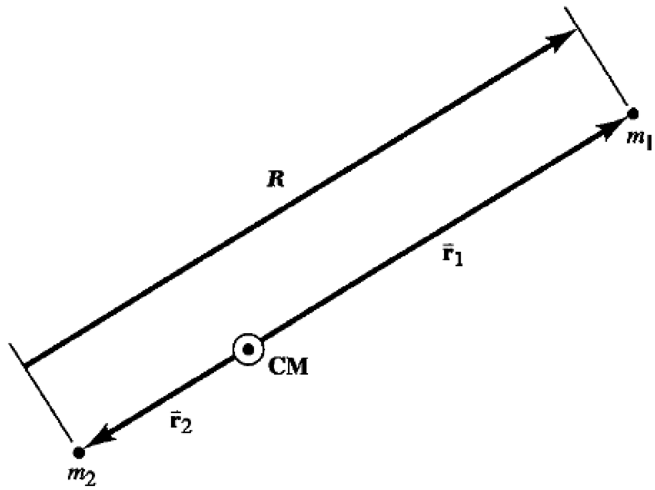

Figure 7.3.1 The relative position vector **R** for the two-body problem.

Substituting this in the equation of motion, we find

$$\ddot{\boldsymbol{R}} = -\frac{G(m_1 + m_2)}{R^2}\boldsymbol{e}_R. \tag{7.25}$$

This is the same as the equation of motion found in the previous section, except that M (the mass of the Sun) has been replaced by $m_1 + m_2$. The motion is therefore an ellipse (or circle, parabola or hyperbola).

Since r_1 and r_2 are proportional, and their sum is R, in the centre-of-mass frame, each mass moves in an elliptical orbit with the centre of mass at a focus. The other mass moves correspondingly, in an elliptical orbit that has the same eccentricity but with a different semi-major axis a_2. It is easy to see that $a_1 + a_2 = a$, where a is the semi-major axis of the ellipse traced by $\boldsymbol{R}$.

Kepler's third law is now replaced by the more general form

$$T^2 = \frac{4\pi^2 a^3}{G(m_1 + m_2)}. \tag{7.26}$$

7.6 Collisions

When two objects collide, they are subject to an internal force that changes the momentum of each. However the total momentum of the two objects remains the same. If we denote the momenta after the collision with a prime, then

$$\boldsymbol{p}_1 + \boldsymbol{p}_2 = \boldsymbol{p}_1' + \boldsymbol{p}_2'. \tag{7.27}$$

The energy need not be conserved during the collision. For example, some of the kinetic energy may be converted into heat during the collision. We can write

$$\frac{\boldsymbol{p}_1^2}{2m_1} + \frac{\boldsymbol{p}_2^2}{2m_2} = \frac{\boldsymbol{p}_1'^2}{2m_1} + \frac{\boldsymbol{p}_2'^2}{2m_2} + Q. \tag{7.28}$$

where Q represents the energy gained or lost in the collision.

If $Q = 0$ (no change in energy) the collision is ***elastic***. If $Q > 0$ (energy is lost), the collision is ***exoergic***. If $Q < 0$ (energy is gained), the collision is ***endoergic***.

For a direct (head-on) collision, in which the motion is one dimensional, one can define the ***coefficient of restitution***, e, which is the ratio of the relative velocities after and before the collision,

$$e = \frac{|v_2' - v_1'|}{|v_2 - v_1|}. \tag{7.29}$$

It is easy to show that

$$Q = \frac{1}{2}\mu v^2(1 - e^2) \tag{7.30}$$

where $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass and $v = |v_2 - v_1|$ is the relative velocity before the collision.

Collisions are simplest in the centre of mass frame. In this frame the initial and final values of the total momentum are both zero,

$$\bar{\boldsymbol{p}}_1 + \bar{\boldsymbol{p}}_2 = \bar{\boldsymbol{p}}_1' + \bar{\boldsymbol{p}}_2' = 0. \tag{7.31}$$

As an example, consider the collision of a moving object m_1 with a stationary object m_2, as seen in the laboratory frame. After the collision, we see the two objects moving away at angles ϕ_1 and ϕ_2 from the initial path of m_1. We wish to determine the final velocities of the two particles and the energy released or absorbed in the collision.

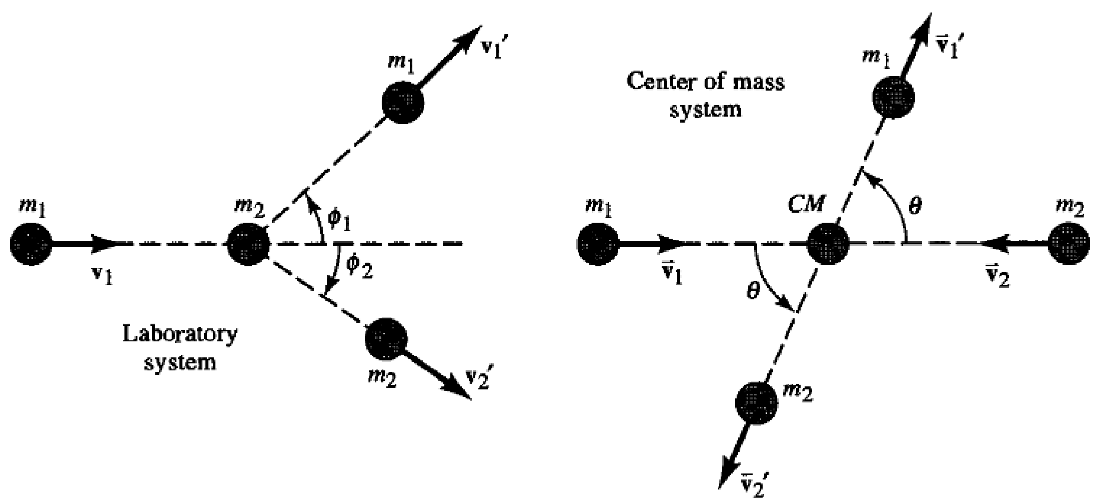

Figure 7.6.1 Comparison of laboratory and center of mass coordinates.

We wish to determine Q, which is the difference between the initial and final kinetic energies,

$$Q = \frac{1}{2}m_1 v_1^2 - \left(\frac{1}{2}m_1 v_1'^2 + \frac{1}{2}m_2 v_2'^2\right). \tag{7.32}$$

To determine this we need to find the final velocities. These are related to the initial velocity by conservation of momentum. The components of the momentum in direction parallel and perpendicular to the initial velocity must be the same before and after the collision. Referring to the figure we see that

$$m_1 v_1 = m_1 v_1' \cos\phi_1 + m_2 v_2' \cos\phi_2, \tag{7.33}$$

$$0 = m_1 v_1' \sin\phi_1 - m_2 v_2' \sin\phi_2, \tag{7.34}$$

We can use these two equations to eliminate v_1' and v_2'. Solving the second equation for v_2', we get

$$m_2 v_2' = m_1 v_1' \frac{\sin\phi_1}{\sin\phi_2}. \tag{7.35}$$

Substituting this in the first equation gives

$$m_1 v_1 = m_1 v_1' \left(\cos\phi_1 + \frac{\sin\phi_1}{\sin\phi_2}\cos\phi_2\right) = m_1 v_1' \sin\phi_1(\cot\phi_1 + \cot\phi_2). \tag{7.36}$$

Finally, we can substitute these results into the equation for Q to get

$$\begin{aligned} Q &= \frac{1}{2}m_1 v_1^2 - \frac{1}{2}m_1 v_1'^2 \left(1 + \frac{m_1}{m_2}\frac{\sin^2\phi_1}{\sin^2\phi_2}\right), \\ &= \frac{1}{2}m_1 v_1^2 \left[1 - \frac{1}{\sin^2\phi_1(\cot\phi_1 + \cot\phi_2)^2}\left(1 + \frac{m_1}{m_2}\frac{\sin^2\phi_1}{\sin^2\phi_2}\right)\right], \qquad (7.37) \\ &= \frac{1}{2}m_1 v_1^2 \left[1 - \frac{m_2 \csc^2\phi_1 + m_1 \csc^2\phi_2}{m_2(\cot\phi_1 + \cot\phi_2)^2}\right]. \qquad (7.38) \end{aligned}$$

7.7 Motion involving a variable mass

So far we have assumed that the masses of objects are constant. But this may not always be true. Rockets, for example, eject part of their mass as exhaust gasses in order to provide propulsion. Raindrops increase collide with smaller droplets as they fall, growing in size. Let's now examine such situations.

Suppose that we have a mass $m(t)$ moving with velocity $\boldsymbol{v}(t)$, which at time t collides with a small mass Δm moving with velocity u and absorbs it. A short time Δt after the collision we have a single object moving with speed $v + \Delta v$.

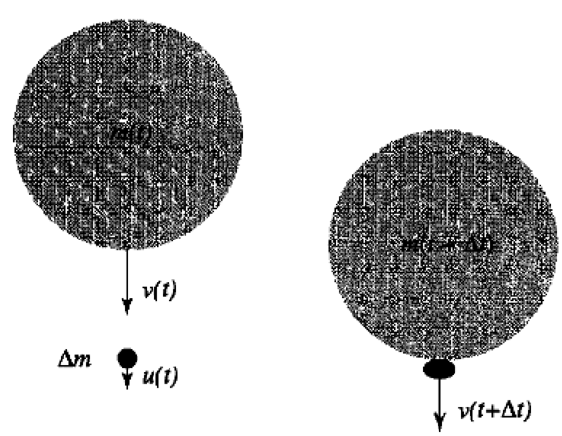

Figure 7.7.1 A mass m gathering up a small mass Δm as it moves through a medium.

The change in the total momentum is

$$\Delta \boldsymbol{p} = (m + \Delta m)(\boldsymbol{v} + \Delta \boldsymbol{v}) - (m\boldsymbol{v} + \Delta m\, \boldsymbol{u}) = m\Delta \boldsymbol{v} + \Delta m(\boldsymbol{v} - \boldsymbol{u}) + \Delta m \Delta \boldsymbol{v}. \tag{7.39}$$

Dividing by Δt and taking the limit $\Delta t \to 0$, we get

$$\frac{d\boldsymbol{p}}{dt} = m\frac{d\boldsymbol{v}}{dt} + \frac{dm}{dt}(\boldsymbol{v} - \boldsymbol{u}). \tag{7.40}$$

This is the rate of change of the total momentum, which must equal the net external force on the system

$$\boldsymbol{F}_{\text{ext}} = m\frac{d\boldsymbol{v}}{dt} + \frac{dm}{dt}(\boldsymbol{v} - \boldsymbol{u}). \tag{7.41}$$

Note that if $\boldsymbol{u}$ is constant, this can be written as

$$\boldsymbol{F}_{\text{ext}} = \frac{d}{dt}[m(\boldsymbol{v} - \boldsymbol{u})], \quad (\text{constant } \boldsymbol{u}). \tag{7.42}$$

This same equation applies to a rocket, except in that case dm/dt is negative. The second term can be written as $-\dot{m}(\boldsymbol{u} - \boldsymbol{v}) = -\dot{m}\boldsymbol{V}$, where $\boldsymbol{V}$ is the relative velocity of the exhaust. $\boldsymbol{F}_{\text{thrust}} = \dot{m}\boldsymbol{V}$ is called the ***thrust*** of the rocket. We see that it is proportional to the relative velocity of the exhaust and the rate at which mass is being expelled. Thus

$$m\frac{d\boldsymbol{v}}{dt} = \boldsymbol{F}_{\text{ext}} + \boldsymbol{F}_{\text{thrust}}. \tag{7.43}$$

As a simple example, consider a rocket in deep space, not subject to any external forces, and suppose that $\boldsymbol{V}$ is constant. Let v_0 and m_0 be the initial speed and mass of the rocket. We wish to find the speed as a function of time.

The problem is one dimensional,

$$m\frac{dv}{dt} = -V\frac{dm}{dt}. \tag{7.44}$$

Multiply by dt and then separating the variables v and m to get

$$dv = -V\frac{dm}{m}. \tag{7.45}$$

Integrating,

$$\int_{v_0}^{v} dv = -V\int_{m_0}^{m} \frac{dm}{m}, \tag{7.46}$$

therefore

$$v = v_0 + V \ln\left(\frac{m_0}{m}\right). \tag{7.47}$$

This shows that to achieve a high velocity, one needs a very high exhaust velocity, and a large mass of propellant.

If the burn rate $\dot{m}$ is constant, $m = m_0 - \dot{m}t$ and the velocity is

$$v(t) = v_0 - V \ln\left(\frac{m_0 - \dot{m}t}{m_0}\right) = v_0 - V \ln\left(1 - \frac{\dot{m}t}{m_0}\right). \tag{7.48}$$

8 Rigid bodies

We can consider rigid bodies as an extension of a multi-particle system, in which the relative positions of all the particles are fixed. Define the ***mass density*** $\rho(\boldsymbol{r})$ to be the mass per unit volume. It is a function of position. The mass contained within an infinitesimal volume dV located at position $\boldsymbol{r}$ is therefore $dm(\boldsymbol{r}) = \rho(\boldsymbol{r})dV$.

8.1 Centre of mass

The centre of mass of the body can be found by replacing sums by integrals,

$$\boldsymbol{r}_{\text{cm}} = \frac{1}{m}\int \boldsymbol{r}\rho(\boldsymbol{r})dV. \tag{8.1}$$

where

$$m = \int \rho(\boldsymbol{r})dV \tag{8.2}$$

is the total mass of the body.

If the body is symmetric about any plane, the centre of mass will lie in that plane. The text book gives some examples of how to calculate the position of the centre of mass for various shapes in Chapter 8.

8.2 Kinetic energy of rotation

Suppose that a body is constrained to rotate about a fixed axis (which may or may not pass through the body). What is the rotational kinetic energy?

In an inertial frame, set up a Cartesian coordinate system in which the z axis coincides with the rotation axis. Let $\boldsymbol{\omega}$ denote the angular velocity vector. The velocity of an element of mass $dm(\boldsymbol{r})$ is given by

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r} = \omega\boldsymbol{k} \times \boldsymbol{r}, \tag{8.3}$$

The velocity components are therefore

$$v_x = -\omega r_y, \tag{8.4}$$

$$v_y = \omega r_x, \tag{8.5}$$

so the kinetic energy of this element is

$$dT = \frac{1}{2}v^2 dm = \frac{\omega^2}{2}(x^2 + y^2)dm \tag{8.6}$$

The total kinetic energy of rotation is obtained by integrating this over the entire body,

$$T = \frac{\omega^2}{2}\int (x^2 + y^2)\rho dV \tag{8.7}$$

We can write this result as

$$T = \frac{1}{2}I_z\omega^2, \tag{8.8}$$

where

$$I_z = \int (x^2 + y^2) \rho dV \tag{8.9}$$

is the ***moment of inertia*** about the z axis.

For a freely rotating body, the axis of rotation will pass through the centre of mass.

8.3 Angular momentum

Let's now consider the rotational angular momentum. For an element dm the z component of the angular momentum is

$$dL_z = (\boldsymbol{r} \times \boldsymbol{v})\, dm = (xv_y - yv_x)\, dm = \omega(x^2 + y^2)\, dm. \tag{8.10}$$

Therefore the total angular momentum about the z axis is

$$L_z = \omega \int (x^2 + y^2) \rho dV = I_z \omega. \tag{8.11}$$

If a torque is applied about the z axis, the angular momentum will change. We previously found that

$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{N} \tag{8.12}$$

For a rigid body, I_z is constant, so the z component of this equation gives us

$$I_z \frac{d\omega}{dt} = N_z. \tag{8.13}$$

Note the following correspondences between linear motion and rotation,

Translation along x axis		Rotation about z axis	
Linear momentum	$p_x = mv_x$	Angular momentum	$L_z = I_z\omega$
Force	$F_x = m\dot{v}_x$	Torque	$N_z = I_z\dot{\omega}$
Kinetic energy	$T = \frac{1}{2}mv^2$	Kinetic energy	$T = \frac{1}{2}I_z\omega^2$

(8.14)

We see that for rotation, I_z plays a similar role as m does for linear motion, with the linear velocity v being replaced by the angular velocity ω.

8.4 The parallel-axis theorem

If the axis of rotation does not go through the centre of mass of the body, we can write

$$x = x_{\text{cm}} + \bar{x}, \tag{8.15}$$

$$y = y_{\text{cm}} + \bar{y}, \tag{8.16}$$

where the bar indicates the centre of mass frame. The moment of inertia about the z axis then becomes

$$\begin{aligned} I_z &= \int (x^2 + y^2) \rho dV \\ &= \int (\bar{x}^2 + \bar{y}^2) \rho dV + (x_{\text{cm}}^2 + y_{\text{cm}}^2) \int \rho dV + 2x_{\text{cm}} \int \bar{x} \rho dV + 2y_{\text{cm}} \int \bar{y} \rho dV, \\ &= I_{\text{cm}} + ml^2, \end{aligned} \tag{8.17}$$

where $l^2 = x_{\text{cm}}^2 + y_{\text{cm}}^2$ is the perpendicular distance from the axis of rotation to the centre of mass. (The last two terms are zero due to the definition of the centre of mass.)

8.5 The radius of gyration

If we divide the moment of inertia by the mass of the object, the result is

$$\frac{I_z}{m} = \frac{1}{m}\int (x^2 + y^2)\rho dV \tag{8.18}$$

which is a density-weighted mean square distance from the rotation axis. The square root of this quantity is a length k, called the radius of gyration,

$$k = \sqrt{\frac{I}{m}} \tag{8.19}$$

If the density is constant, the radius of gyration depends only on the shape and size of the object. Specifying k, and the axis, is equivalent to specifying the moment of inertia per unit mass. Some values are listed in Table 8.3.1 in the text book.

For example, a thin rod of length a rotating about its centre has $k^2 = a^2/12$, a uniform sphere of radius a has $k^2 = 2a^2/5$ and solid cylinder rotating about its axis has $k^2 = a^2/2$.

8.6 Example: object rolling down an inclined plane

As an example, consider the system shown in the diagram below

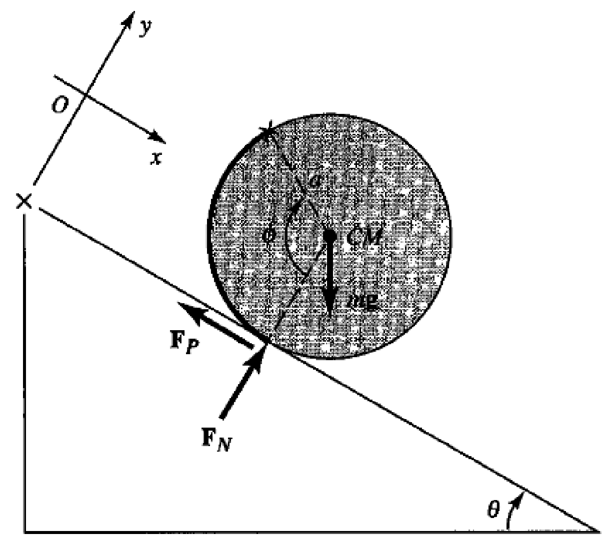

Figure 8.6.1 Body rolling down an inclined plane.

The forces acting on the body, through the centre of mass (CM) are gravity $m\boldsymbol{g}$, a normal force $\boldsymbol{F}_N$ and a frictional force $\boldsymbol{F}_P$. Choose Cartisian (x, y) coordinates as shown. In these coordinates, the equations for the linear motion of the CM are

$$m\ddot{x} = mg\sin\theta - F_P, \tag{8.20}$$
$$m\ddot{y} = mg\cos\theta + F_N. \tag{8.21}$$

As the object moves, the y coordinate of the CM does not change, therefore $\ddot{y} = 0$, so the second equation tells us that the normal force is

$$F_N = -mg\cos\theta \tag{8.22}$$

The first equation tells us about the motion in the x direction, but we cannot solve it until we determine F_P. To find it we observe that this force will exert a torque $\boldsymbol{r} \times \boldsymbol{F}_P$ about the CM of the object. The magnitude of the torque is

$$N = aF_p.$$

This will cause the object to rotate,

$$N = aF_p = I\dot{\omega}$$

where $\omega = \dot{\phi}$ is the angular speed and I is the moment of inertia of the object about the axis of rotation.

We now have and expression for F_P in terms of ω, but not x. To relate these two quantities, observe that if the CM is moving with speed $\dot{x}$ and there is no slipping, the angular speed of rotation is

$$\omega = \frac{\dot{x}}{a}.$$

Therefore,

$$\dot{\omega} = \frac{\ddot{x}}{a}.$$

We can now substitute these results into the equation for the acceleration,

$$m\ddot{x} = mg\sin\theta - \frac{I}{a^2}\ddot{x}.$$

Solving this for $\ddot{x}$, we get

$$\ddot{x} = \frac{g\sin\theta}{1 + I/ma^2} = \frac{g\sin\theta}{1 + k^2/a^2}.$$

For a cylinder, $k^2 = a^2/2$ so

$$\ddot{x} = \frac{2}{3}g\sin\theta.$$

For a sphere, $k^2 = 2a^2/3$ so

$$\ddot{x} = \frac{5}{7}g\sin\theta.$$

We see that the resulting acceleration is independent of the mass and the radius of the object, and that a sphere will accelerate faster than a cylinder.

9 Lagrangian mechanics

An alternative approach to Newtonian mechanics was developed by Leibniz, Bernoulli, D'Alembert, Lagrange and Hamilton, not long after Newton. A key difference of this formalism is its reliance on scalar quantities, such as energy, as opposed to the vectors of the Newtonian approach. This approach is often simpler, particularly for complex problems. It also provides new insight into dynamical systems and let to the discovery of an extraordinary result, *Hamilton's principle*.

9.1 Generalized coordinates

We have used several coordinate systems to describe the motions of objects. For example, a pendulum swinging in a plane can be described by Cartesian coordinates (x, y) both of which would be functions of time. However, it is simpler in this case to use polar coordinates (r, θ) because, since r is constant, only one variable (θ) is needed to describe the position of the pendulum.

Generalized coordinates q_i are any set of independent scalar coordinates that just suffice to uniquely specify the configuration of a system at any particular time.

The number of generalized coordinates needed to specify the configuration of a system is the same as the number of ***degrees of freedom*** of the system.

For example, consider two masses attached to the end of a thin, massless rigid rod of length d. The masses are both free to move in any direction subject to the ***constraint*** that the distance between them is constant and equal to d. If there was no rod, we would need six coordinates (in three dimensions) to specify the positions of the two masses. But with the rod, these coordinates are not all independent because of the constraint

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 - d^2 = 0. \tag{9.1}$$

Because of the constraint, the number of degrees of freedom is reduced by one, from 6 to 5. Therefore, the system can be described by five generalized coordinates. These could be taken to be, for example, the coordinates $x_{\text{cm}}, y_{\text{cm}}, z_{\text{cm}}$ of the centre of mass of the system, and polar coordinates θ, ϕ of one of the objects with respect to the centre of mass. These five generalized coordinates suffice to uniquely specify the positions of both masses.

In general, if we have N particles in three-dimensional space, connected by m independent constraints, there will exist $3N - m$ generalized coordinates.

Each constraint can be expressed in the form

$$f_j(x_i, y_i, z_i, t) = 0 \tag{9.2}$$

for some functions f_j $(j = 1, \ldots, m)$. Constraints of this form are said to be ***holonomic***.

An example of a non-holonomic constraint would be confining a particle to the exterior of a sphere,

$$x^2 + y^2 + z^2 - R^2 \geqslant 0. \tag{9.3}$$

Such a constraint does not reduce the number of degrees of freedom (one still needs three coordinates to specify the location of the particle).

Given generalized coordinates q_i $(i = 1, \ldots, n)$ we can define ***generalized velocities***

$$\dot{q}_i = \frac{dq_i}{dt}. \tag{9.4}$$

9.2 Kinetic and potential energy in generalized coordinates

The potential energy of the system is a function of the coordinates of the particles, and therefore can be written as some function $V(q_i)$ of the generalized coordinates.

Similarly, the kinetic energy can be written as a function of the generalized coordinates and the generalized velocities, $T(q_i, \dot{q}_i)$.

As an example, consider a small mass m moving in the gravitational field of a much larger mass M that is at rest. For the generalized coordinates we take (r, θ, ϕ) of a spherical coordinate system centred on the mass. The potential energy is then

$$V(r) = -\frac{GMm}{r},$$

which is a function of the single generalized coordinate r.

In Section 1 we found that

$$\boldsymbol{v} = \dot{r}\,\boldsymbol{e}_r + r\dot{\phi}\sin\theta\,\boldsymbol{e}_\phi + r\dot{\theta}\,\boldsymbol{e}_\theta,$$

so the kinetic energy is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\dot{r}^2 + r^2\sin^2\theta\dot{\phi}^2 + r^2\dot{\theta}^2\right).$$

This is a function of the generalized coordinates and the generalized velocities, and is *quadratic* in the latter (powers of the velocities are no greater than 2).

9.3 Hamilton's variational principle

We have seen that the configuration of a system can be described by some number n of generalized coordinates which are functions of time t. These coordinates span an n-dimensional ***configuration space***. Every point in configuration space specifies a unique the state of the system.

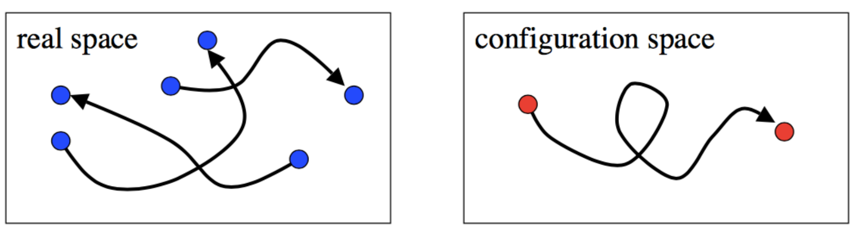

Figure 9.1: Configuration space. Credit: Masahiro Morii, Harvard

As the system evolves dynamically from some initial state at time t_1 to a later state at time t_2, it follows a unique path $q_i(t)$ through configuration space from $q_i(t_1)$ to $q_i(t_2)$. One could determine the path by applying Newton's laws, but in 1834 Hamilton discovered a different approach following from his work on the unification of optics and mechanics.

The ***action*** J is defined by the integral

$$J(t_1, t_2) = \int_{t_1}^{t_2} L dt$$

where $L = T - V$ is the ***Lagrangian*** of the system. ***Hamilton's principle*** states that *the path followed by the system is the one that either maximizes or minimizes the action.* Mathematically this is expressed as

$$\delta J = \delta \int_{t_1}^{t_2} L dt = 0.$$

Here δ represents the change due to an infinitesimal ***variation*** of the the path along which the integral is evaluated. The path can be varied in an arbitrary manner, provided that it starts and ends at the same points. In other words the variations must vanish at the end points.

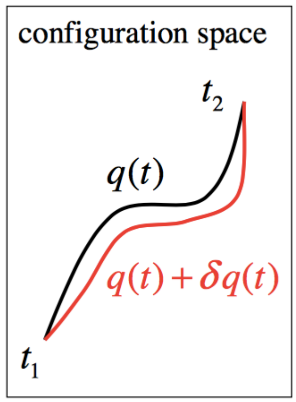

Figure 9.2: A path variation. Credit: Masahiro Morii, Harvard

In general, the Lagrangian will depend both on q_i and $\dot{q}_i$, so we must consider variations in $\dot{q}_i$ as well as q_i. In other words, the *rate* at which the system moves through configuration space may vary, even if the path is the same.

Another way to think of this is to imagine a path in $2n$-dimensional phase space, spanned by q_i and $\dot{q}_i$. Variations in $\dot{q}_i$ correspond to changing the path in the generalized velocity dimensions.

9.4 Lagrange's equations

Let's see what the implications of Hamilton's principal are. We wish to determine under what conditions the principal may be satisfied. We begin by rewriting the principle, making it more clear what the functional dependencies are,

$$\delta \int_{t_1}^{t_2} L[q_i(t), \dot{q}_i(t)] dt = 0. \tag{9.5}$$

The left side is the change of the integral, resulting from some variation of the path $\delta q_i(t)$. Now the integral can be thought of as an infinite sum, and the change of the sum is the sum of the changes. Therefore, we can take the δ symbol inside the integral,

$$\int_{t_1}^{t_2} \delta L[q_i(t), \dot{q}_i(t)] dt = 0. \tag{9.6}$$

The integrand now is the change in L due to the variation of the path. That can be split into two parts, since L depends both on q_i and on $\dot{q}_i$. For each part we must sum over possible variations in each of the n dimensions,

$$\int_{t_1}^{t_2} \left[\sum_i \frac{\partial L}{\partial q_i} \delta q_i + \sum_i \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right] dt = 0. \tag{9.7}$$

Now in the second term we have $\delta \dot{q}_i$. This is a variation of the rate at which the system moves along the path q_i. It is therefore the difference between two slightly different functions of time $q_i(t)$ that follow the same path at different rates. In other words,

$$\delta \dot{q}_i = \frac{d}{dt} \delta q_i. \tag{9.8}$$

The second term in the integrand can now be written in the form

$$\sum_i \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} \delta q_i = \frac{d}{dt} \left(\sum_i \frac{\partial L}{\partial \dot{q}_i} \delta q_i \right) - \sum_i \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i. \tag{9.9}$$

The first term on the right side is a total derivative, so its integral is just the difference between the values at the end points,

$$\int_{t_1}^{t_2} \left[\sum_i \frac{\partial L}{\partial q_i} \delta q_i - \sum_i \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \right] dt + \left[\sum_i \frac{\partial L}{\partial \dot{q}_i} \delta q_i \right]_{t_1}^{t_2} = 0. \tag{9.10}$$

This last term is in fact zero because the variation δq_i is zero at the end points. Factoring out common terms in the integral, we get

$$\int_{t_1}^{t_2} \sum_i \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right] \delta q_i dt = 0. \tag{9.11}$$

Now, this must be true for *any* small variation δq_i, otherwise the path would not be an extremum of the action. That can only happen if the quantity in square brackets is exactly zero.

We thus arrive at ***Lagrange's equations***,

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0. \tag{9.12}$$

These are n partial differential equations that describe the motion of the system.

9.5 Examples of Lagrangian mechanics

9.5.1 Harmonic oscillator

Consider a one-dimensional harmonic oscillator. We need only one generalized coordinate, x, the displacement of the oscillator. The Lagrangian is

$$L = T - V = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2 \tag{9.13}$$

There is only one Lagrange equation,

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = kx - \frac{d}{dt} m \dot{x} = 0. \tag{9.14}$$

which gives us the equation of motion that we previously derived from Newtonian mechanics,

$$m\ddot{x} - kx = 0. \tag{9.15}$$

9.5.2 Particle in a central force field

Consider now a particle moving in the plane $z = 0$ subject to a central force f(r). For the generalized coordinates we take $q_1 = r$ and $q_2 = \theta$. These are related to the Cartesian coordinates x and y by the relations

$$x = r\cos\theta, \tag{9.16}$$

$$y = r\sin\theta. \tag{9.17}$$

The velocities are

$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta, \tag{9.18}$$

$$\dot{y} = \dot{r}\sin\theta + r\dot{theta}\cos\theta. \tag{9.19}$$

The kinetic energy is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \tag{9.20}$$

so the Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r). \tag{9.21}$$

Now calculate the partial derivatives,

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{\partial V}{\partial r} = mr\dot{\theta}^2 + f(r), \qquad \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \tag{9.22}$$

$$\frac{\partial L}{\partial \theta} = 0, \qquad \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}. \tag{9.23}$$

The two Lagrange equations give

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r} \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}, \tag{9.24}$$

$$m\ddot{r} = mr\dot{\theta}^2 + f(r), \qquad \frac{d}{dt}(mr^2\dot{\theta}) = 0. \tag{9.25}$$

Observe that the last equation tells us that $mr^2\dot{\theta}$, the orbital angular momentum of the particle, is a constant of the motion. This happened because the Lagrangian does not contain θ. This is an example of ***Noether's theorem*** (we will discuss this later).

9.5.3 Atwood's machine

Suppose that two masses m_1 and m_2 are attached to a massless string that passes over a pulley, as shown in Figure 10.5.1. Let the length of the string be l and the radius of the pulley be a. A single coordinate x is sufficient to specify the position of the system.

The kinetic energy is

$$T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}I\frac{\dot{x}^2}{a^2}, \tag{9.26}$$

and the potential energy is

$$V = -m_1 g x - m_2 g(l - \pi a - x) \tag{9.27}$$

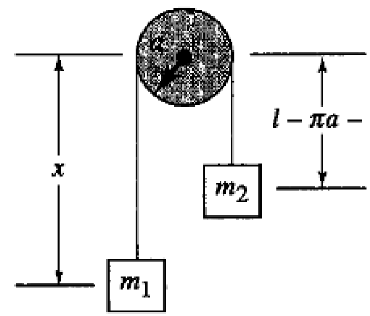

Figure 10.5.1 An Atwood machine.

which gives the Lagrangian

$$L = \frac{1}{2}\left(m_1 + m_2 + \frac{I}{a^2}\right)\dot{x}^2 + (m_1 - m_2)gx + m_2 g(l - \pi a). \tag{9.28}$$

The single Lagrange equation becomes

$$\left(m_1 + m_2 + \frac{I}{a^2}\right)\ddot{x} = (m_1 - m_2)g \tag{9.29}$$

which tells us that the acceleration is

$$\ddot{x} = \frac{(m_1 - m_2)g}{m_1 + m_2 + I/a^2}. \tag{9.30}$$

9.5.4 Particle sliding on a moveable inclined plane

Consider a mass m sliding down a frictionless inclined plane of angle θ. The inclined plane is rests on a frictionless horizontal surface and is free to slide horizontally on this plane. Find the equations of motion.

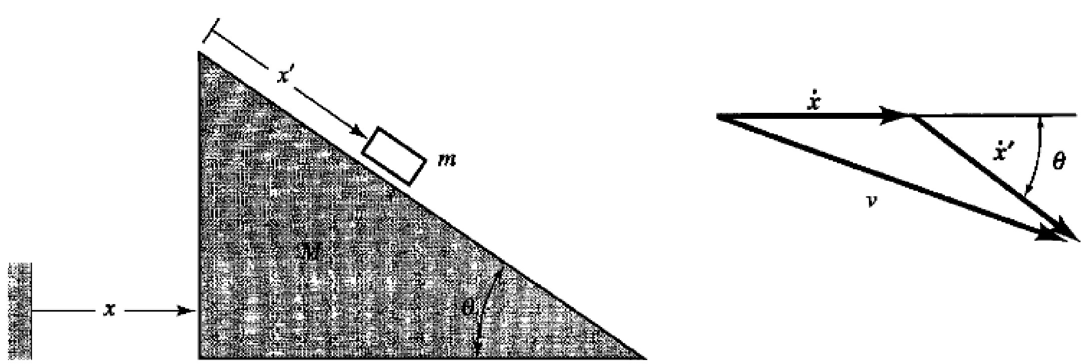

Figure 10.5.3 A block sliding down a movable wedge or inclined plane.

There are two degrees of freedom, which can be represented by the generalized coordinates $q_1 = x$ and $q_2 = x'$, where x is the horizontal position of the inclined plane and x' is the distance along the plane to the mass, measured from the top.

The kinetic energy is the sum of the kinetic energy of m and M. The velocity v of m can be found either by vector analysis, or by the cosine rule,

$$v^2 = (\dot{\boldsymbol{x}} + \dot{\boldsymbol{x}}') \cdot (\dot{\boldsymbol{x}} + \dot{\boldsymbol{x}}') = \dot{x}^2 + \dot{x}'^2 + 2\dot{x}\dot{x}' \cos(\theta). \tag{9.31}$$

The potential energy is $-mgx' \sin\theta$ (we need not include the potential energy of M as it does not change). Therefore, the Lagrangian is

$$L = T - V = \frac{1}{2}[M\dot{x}^2 + m(\dot{x}^2 + \dot{x}'^2 + 2\dot{x}\dot{x}' \cos\theta)] + mgx' \sin\theta. \tag{9.32}$$

There are two Lagrange equations,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}'} = \frac{\partial L}{\partial x'}, \tag{9.33}$$

$$\frac{d}{dt}\big[(M + m)\dot{x} + m\dot{x}' \cos\theta\big] = 0, \qquad \frac{d}{dt}\big(\dot{x}' + \dot{x}\cos\theta\big) = g\sin\theta. \tag{9.34}$$

We see that $(M + m)\dot{x} + m\dot{x}' \cos\theta$ is a constant of the motion. This is a consequence of the Lagrangian being independent of x'. Evaluating the time derivatives,

$$(M + m)\ddot{x} + m\ddot{x}' \cos\theta = 0, \qquad \ddot{x}' + \ddot{x}\cos\theta = g\sin\theta. \tag{9.35}$$

Solving these for $\ddot{x}$ and $\ddot{x}'$, we find

$$\ddot{x} = \frac{-g\sin\theta\cos\theta}{(m + M)/m - \cos^2\theta}, \qquad \ddot{x}' = \frac{g\sin\theta}{1 - m\cos^2\theta/(m + M)}. \tag{9.36}$$

9.5.5 The physical pendulum

A real pendulum is not a point mass. The mass is extended and so one needs to consider rotational energy of the mass and the pendulum rod. Let the moment of inertial of the pendulum about the suspension point be I. There is only one degree of freedom, so one coordinate is needed. We can take this to be θ, the angle between the pendulum arm and the vertical axis. The kinetic energy is $I\dot{\theta}^2/2$ and the potential energy is $mgl(1 - \cos\theta)$, where l is the distance between the suspension point and the centre of mass.

Therefore, the Lagrangian is

$$L = \frac{1}{2}I\dot{\theta}^2 - mg(1 - \cos\theta). \tag{9.37}$$

The Lagrange equation is

$$I\ddot{\theta} + mgl\sin\theta = 0 \tag{9.38}$$

For small oscillations, $\sin\theta \simeq \theta$ and this is a harmonic oscillator equation. The period is therefore

$$T = 2\pi\sqrt{\frac{I}{mgl}}. \tag{9.39}$$

9.6 Non-conservative and dissipative forces

So far we have considered only forces that can be derived from a potential. Can Lagrangian mechanics handle non-conservative or dissipative forces? Yes, one can include such forces by adding additional terms to the Lagrangian. However, the simplest way to deal with such forces is to just add them to the Lagrange equations.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i(q_i, \dot{q}_i), \tag{9.40}$$

where Q_i represents the non-conservative or dissipative forces affecting the coordinate q_i.

For example, consider a large molecule moving in a solution under the action of a constant force F_0 acting in the x direction. Resisting the motion is a drag force $\boldsymbol{Q} = -c\boldsymbol{v}$ that is proportional to the velocity of the molecule. We wish to find the equation of motion.

The problem is one-dimensional, with a single generalized coordinate $q_1 = x$. The potential that gives rise to the constant force is

$$V(x) = -F_0 x, \tag{9.41}$$

so the Lagrangian is

$$L = \frac{1}{2}m\dot{x}^2 + F_0 x, \tag{9.42}$$

and the additional non-conservative force is $Q_1 = -c\dot{x}$. The Lagrange equation (9.40) becomes

$$m\ddot{x} - F_0 = -c\dot{x} \tag{9.43}$$

which is the same as the Newtonian result.

9.7 Conjugate momenta

For each generalized coordinate q_i, we can define a generalized momentum p_i that is associated with it, according to

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{9.44}$$

The various p_i are called ***conjugate momenta***.

For example, a free particle of mass m has the Lagrangian

$$L = \frac{1}{2}m\dot{x}^2. \tag{9.45}$$

The momentum conjugate to the generalized coordinate x is therefore

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \tag{9.46}$$

which is the usual linear momentum.

Now consider a wheel having moment of inertia I. Let θ be the rotation angle of the wheel. The Lagrangian is

$$L = \frac{1}{2}I\dot{\theta}^2, \tag{9.47}$$

so the momentum conjugate to θ is

$$p_\theta = I\dot{\theta} = I\omega. \tag{9.48}$$

This quantity is the *angular* momentum of the wheel.

In terms of the conjugate momenta, the Lagrange equations become

$$\frac{dp_i}{dt} = \frac{\partial L}{\partial q_i}. \tag{9.49}$$

From this we see that if the Lagrangian does not contain a particular generalized coordinate q_k, then the corresponding generalized momentum is conserved,

$$\frac{dp_k}{dt} = 0. \tag{9.50}$$

We then say that q_k is an ***ignorable coordinate*** and its conjugate momentum is a ***constant of the motion***.

9.8 Constraints

In the examples so far, we have implicitly used any constraints present to reduce the number of degrees of freedom of the problem. That is usually the simplest way to solve the problem. However, sometimes it may be easier to include all the coordinates, and relate them with equations of constraint. This would be the case if one wanted to determine the forces associated with the constraints (such as the tension in a pendulum wire). To solve the problem this way, we use the method of ***Lagrange multipliers***.

To illustrate the method, suppose that we have a system that has two generalized coordinates q_1 and q_2 related by a holonomic constraint that can be written as

$$f(q_1, q_2) = 0 \tag{9.51}$$

for some function f.

Hamilton's principle tells us that

$$\delta \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} \sum_{i=1}^{2} \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right] \delta q_i dt = 0 \tag{9.52}$$

Previously we argued that because the δq_1 were all arbitrary functions of time, this equation would be satisfied only if the quantity in square brackets was identically zero for all values of i. But now we cannot do that because the two coordinates q_1 and q_2 are related. Therefore, their variations are related also.

To find the relationship between δq_1 and δq_2, consider the variation of the constraint equation,

$$\delta f = \frac{\partial f}{\partial q_1}\delta q_1 + \frac{\partial f}{\partial q_2}\delta q_2 = 0. \tag{9.53}$$

Therefore,

$$\delta q_2 = -\left(\frac{\partial f/\partial q_1}{\partial f/\partial q_2}\right)\delta q_1. \tag{9.54}$$

Hamilton's principle becomes

$$\int_{t_1}^{t_2}\left\{\left[\frac{\partial L}{\partial q_1}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right)\right]-\left[\frac{\partial L}{\partial q_2}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right)\right]\left(\frac{\partial f/\partial q_1}{\partial f/\partial q_2}\right)\right\}\delta q_1 dt = 0 \tag{9.55}$$

Now there is just one δq_i, so the quantity inside the braces must be zero. Thus,

$$\left[\frac{\partial L}{\partial q_1}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right)\right]\left(\frac{\partial f}{\partial q_1}\right)^{-1} = \left[\frac{\partial L}{\partial q_2}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right)\right]\left(\frac{\partial f}{\partial q_2}\right)^{-1}. \tag{9.56}$$

The left and right side of this equation are both functions of time, so they can be equal only if they are the *same* function of time, $-\lambda(t)$ say. Therefore,

$$\left[\frac{\partial L}{\partial q_i}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right]\left(\frac{\partial f}{\partial q_i}\right)^{-1} = -\lambda(t), \tag{9.57}$$

which we can write as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i} = \lambda(t)\frac{\partial f}{\partial q_i} \equiv Q_i(t). \tag{9.58}$$

Comparing this to Equation (9.40), we see that the quantities $Q_i(t)$ act like non-conservative forces. They are in fact the the forces of the constraint.

We now have three unknown functions, $q_1(t)$, $q_2(t)$ and $\lambda(t)$. But we also have three equations: the two Lagrange equations above, and the equation of the constraint. Therefore the problem can be solved.

For example, consider again a wheel rolling down an inclined plane. This time we take two generalized coordinates, x and ϕ. They are not independent, but are related since $x = a\phi$. The constraint equation is therefore

$$f(x,\phi) = x - a\phi = 0,$$

and the Lagrangian is

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\phi}^2 + mgx\sin\theta$$

The Lagrange equations are

$$m\ddot{x} - mg\sin\theta = Q_x = \lambda, \tag{9.59}$$

$$I\ddot{\phi} = Q_\phi = -\lambda a. \tag{9.60}$$

Differentiating the constraint equation we find

$$\ddot{\phi} = \frac{\ddot{x}}{a}. \tag{9.61}$$

Substituting this into the second Lagrange equation and eliminating λ we get

$$\left(m + \frac{I}{a^2}\right)\ddot{x} = mg\sin\theta \tag{9.62}$$

which is the equation of motion.

If instead of eliminating λ we eliminate $\ddot{x}$, we get

$$(ma^2/I + 1)/\lambda = -mg\sin\theta \tag{9.63}$$

so

$$\lambda = -\frac{mg\sin\theta}{a^2/k^2 + 1}. \tag{9.64}$$

For a disk, $k^2 = a^2/2$ so this becomes

$$\lambda = -\frac{1}{3}mg\sin\theta. \tag{9.65}$$

and the equation of motion becomes

$$\ddot{x} = \frac{2}{3}g\sin\theta. \tag{9.66}$$

as we obtained previously using Newton's equations.

The force of constraint is

$$Q_x = -\frac{Q_y}{a} = \lambda = -\frac{1}{3}mg\sin\theta. \tag{9.67}$$

This is the frictional force that is producing the constraint.

9.9 Hamilton's equations

An alternative formulation of mechanics was developed by William Rowan Hamilton in 1833, building on Lagrangian mechanics. Hamilton's approach describes a mechanical system using generalized coordinates $q_i(t)$ and generalized momenta $p_i(t)$, which are given equal status. The configuration, ***and motion***, of the system at any time t are then described by a single point in a $2n$ dimensional phase space.

Building on Lagrange's description of mechanics, Hamiltonian considered the function

$$H(q_i, p_i, t) = \sum_i \dot{q}_i p_i - L(q_i, \dot{q}_i, t). \tag{9.68}$$

which we call the ***Hamiltonian***. Recall that

$$L = T(q_i, \dot{q}_i, t) - V(q_i) \tag{9.69}$$

where T is generally a homogeneous quadratic function of the $\dot{q}$'s and V is a function of the q's alone. The conjugate momenta are defined by $p_i = \partial L/\partial \dot{q}_i$. The first term in the definition of the Hamiltonian is therefore

$$\sum_i \dot{q}_i p_i = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = \sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = 2T. \tag{9.70}$$

The last step follows from Euler's theorem for homogeneous functions f of degree n in the variables x_i

$$\sum_i x_i \frac{\partial f}{\partial x_i} = nf. \tag{9.71}$$

(For example, if $f = ax_1^2 + bx_2^2 + cx_1x_2$, we find

$$\sum_i x_i \frac{\partial f}{\partial x_i} = x_1(2ax_1) + x_1(2cx_2) + x_2(2bx_2) + x_2(cx_1) = 2f$$

in agreement with Euler's theorem.)

Substituting this result into the definition of the Hamiltonian, we find

$$H = 2T - L = 2T - (T - V) = T + V$$

which is the total energy of the system.

Note: this result holds only if the potential does not depend on any of the velocities $\dot{q}_i$. This is generally, but not always, the case. The definition (Eqn. 9.68) is always correct, but the Hamiltonian is not always equal to the total energy.

Now, H is a function of q_i and p_i, so lets see how it changes under a variation of the path in configuration space,

$$\delta H = \delta \left[\sum_i \dot{q}_i p_i - L \right] = \sum_i \left[p_i \delta\dot{q} + \dot{q}_i \delta p_i - \frac{\partial L}{\partial \dot{q}_i} \delta\dot{q}_i - \frac{\partial L}{\partial q_i} \delta q_i \right] \tag{9.72}$$

The first and third terms cancel, due to the definition of the conjugate momentum. We can also use Lagrange's equations in the form $\dot{p}_i = \partial L / \partial q_i$ to simplify this,

$$\delta H = \sum_i \left[\dot{q}_i \delta p_i - \dot{p}_i \delta q_i \right] \tag{9.73}$$

Now for any function $H(p_i, q_i)$ we must have

$$\delta H = \sum_i \left[\frac{\partial H}{\partial p_i} \delta p_i - \frac{\partial H}{\partial q_i} \delta q_i \right] \tag{9.74}$$

so these two equations will be equal, for arbitrary variations, only if

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}. \tag{9.75}$$

These are ***Hamilton's equations***. They are $2n$ *first-order* differential equations, equivalent to the n *second-order* differential equations of Lagrange.

In Hamiltonian mechanics, p_i and q_i are independent functions of time, on an equal footing. Together they span a $2n$-dimensional *phase space*. Each point in this phase space specifies both the configuration of the system *and* its motion. As the system evolves, it traces a path through phase space, according to Hamilton's equations.

While not particularly advantageous for the solution of problems in mechanics, the Hamiltonian approach provided new insights into the physics and mathematics of mechanics. This formalism is used extensively in quantum mechanics, where the Hamiltonian H is regarded as an *operator* that generates the time evolution of the quantum-mechanical system.

9.10 Examples of Hamiltonian mechanics

9.10.1 Harmonic oscillator

For the one-dimensional harmonic oscillator,

$$T = \frac{1}{2}m\dot{x}^2, \qquad V = \frac{1}{2}kx^2, \qquad L = T - V, \tag{9.76}$$

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad \text{so} \qquad \dot{x} = \frac{p}{m}. \tag{9.77}$$

The Hamiltonian is

$$H = T + V = \frac{p^2}{2m} + \frac{kx^2}{2}, \tag{9.78}$$

and Hamilton's equations give

$$\dot{p} = -\frac{\partial H}{\partial x} = -kx, \tag{9.79}$$

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \tag{9.80}$$

Hence

$$m\ddot{x} + kx = 0. \tag{9.81}$$

9.10.2 Particle moving in a central force field

As before, we use polar coordinates in the plane of the orbit. The Lagrangian is

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r), \tag{9.82}$$

so the conjugate momenta are

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \tag{9.83}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}. \tag{9.84}$$

Substituting these in the expression for the kinetic energy, we get the Hamiltonian

$$H = T + V = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + V(r) \tag{9.85}$$

We can now write down Hamilton's equations. Recalling that $f = -dV/dr$,

$$\dot{p_r} = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} + f(r) \qquad \dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \tag{9.86}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0 \qquad \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}. \tag{9.87}$$

The first equation on the second line tells us that the angular momentum $p_\theta = mr^2\dot{theta} \equiv ml$ is conserved. The first line then gives

$$m\ddot{r} - \frac{ml^2}{r^3} = f(r), \tag{9.88}$$

which is the same equation that we found earlier using Newtonian and Lagrangian methods.

9.11 Legendre transformations

The definition of the Hamiltonian (Eqn. 9.68) is an example of a class of mathematical transformations studied by the mathematician Adrien-Marie Legendre.

Suppose that we have a system described by a function $f(x, y)$ of two variables, x and y. Its differential can be written as

$$df = udx + vdy \tag{9.89}$$

where

$$u = \frac{\partial f}{\partial x}, \qquad v = \frac{\partial f}{\partial y}. \tag{9.90}$$

Suppose now that we wish to change the description from the variables x, y to u, y. This can be done by defining a new function

$$g(u, y) = f(x, y) - ux \tag{9.91}$$

The differential of the function g is then

$$dg = df - udx - xdu = vdy - xdu. \tag{9.92}$$

This has the desired form, which can be compared to

$$dg = \frac{\partial g}{\partial u}du + \frac{\partial g}{\partial y}dy. \tag{9.93}$$

The quantities x and v are therefore related to u and y by

$$x = -\frac{\partial g}{\partial u}, \qquad v = \frac{\partial g}{\partial y}. \tag{9.94}$$

From this we see that the Hamiltonian $H(p, q)$ is the result of applying a Legendre transformation to the Lagrangian $L(\dot{q}, q)$ in order to change the description from $\dot{q}, q$ to p, q. The generalization to higher dimensions $\dot{q}_i, q_i$ is a simple extension.

Legendre transformations are also used in other areas of physics, such as thermodynamics, to change state variables.

9.12 More difficult examples solved using Lagrangian methods

9.12.1 Dropped plank

Two people are holding the ends of a uniform plank of length l and mass m. Show that if one person suddenly lets go, the load supported by the other person suddenly from from $mg/2$ to $mg/4$. Show that the initial downward acceleration of the free end is $3g/2$.

There are no sideways forces, so there will be no sideways motion of the centre of mass of the plank. Therefore we need only two generalized coordinates. Let y be the height of the centre of mass and let θ be the angle that the plank makes with the horizontal, in the sense that θ increases as the end of the plank drops. Initially, $\theta = 0$ and we can take $y = 0$.

The Lagrangian is

$$L = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}I\dot{\theta}^2 - mgy \tag{9.95}$$

There is also a constraint in that, initially, one end of the plank is held at a fixed height (which we have defined to be zero). The height of this end can be written, when $\theta \ll 1$, as $y + l\theta/2$, so our constraint equation is

$$f(y, \theta) = y + \frac{l}{2}\theta = 0 \tag{9.96}$$

Lagrange's equations are

$$m\ddot{y} + mg = \lambda\frac{\partial f}{\partial y} = \lambda, \tag{9.97}$$

$$I\ddot{\theta} = \lambda\frac{\partial f}{\partial \theta} = \frac{l}{2}\lambda. \tag{9.98}$$

Differentiating the constraint equation gives us

$$\ddot{y} = -\frac{l}{2}\ddot{\theta} \tag{9.99}$$

Eliminating λ from the Lagrange equations, we find

$$m\ddot{y} + mg = \frac{2I}{l}\ddot{\theta} = -\frac{4I}{l^2}\ddot{y}, \tag{9.100}$$

so

$$\ddot{y} = -\frac{g}{1 + 4I/ml^2} = -\frac{g}{1 + 4k^2/l^2} \tag{9.101}$$

Referring to Table 8.3.1 in the text, we see that for a thin rod of length l, $k^2 = l^2/12$, so this simplifies to

$$\ddot{y} = -\frac{g}{1 + 4I/ml^2} = -\frac{g}{1 + 1/3} = -\frac{3}{4}g. \tag{9.102}$$

This is the downward acceleration of the centre of mass. The vertical position of the free end is $2y$, so its acceleration is $2\ddot{y} = -3g/2$, namely $3g/2$ in the downward direction.

From the first Lagrange equation we see that the force of constraint, referred to the centre of mass, is $Q_y = \lambda = (-3/4 + 1)mg = mg/4$, which is the upward force that the person holding the end of the plank must exert.

The second Lagrange equation tells us that there is a torque $Q_\theta = \lambda l/2 = mgl/8$. This is the torque produced by the upward force calculated above, $N = Q_y(l/2)$.

9.12.2 Rotating plank

A thin uniform plank of length l lies at rest on a horizontal sheet of ice. If the plank is given a kick at one end in a direction normal to the plank, show that the plank will begin to rotate about a point located a distance $l/6$ from the centre.

This system is confined to a plane, that of the ice sheet, which we assume is frictionless. If we were to kick the plank at its centre (the centre of mass), it would begin to move sideways at constant speed. But because the kick is at one end, there is also a torque, so the plank will also begin to rotate. In the centre-of-mass frame, the plank must rotate about its centre of mass. (If it didn't, the centre of mass would be moving in a circle, which would violate conservation of linear momentum since no forces are acting). The combination of rotation around the centre of mass, and translation of the centre of mass means that there is some point on the plank where the rotational velocity is equal and opposite to the velocity of the centre of mass so the velocity of that point as seen in the lab (ice) frame is zero right after the kick. This is the point that we wish to find.

We can describe the position of the plank by giving x, y coordinates of the centre of mass, and a rotation angle θ, with respect to the x axis, that is initially zero. Since the initial impulse is in the y direction, there are no components of force in the x direction. So the x coordinate of the centre of mass will not change. Therefore, we need only two generalized coordinates, y and θ.

We can now write down the Lagrangian,

$$L = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}I\dot{\theta}^2. \tag{9.103}$$

There is no potential energy in this case.

The Lagrange equations give

$$m\ddot{y} = Q_y(t), \tag{9.104}$$

$$I\ddot{\theta} = Q_\theta(t). \tag{9.105}$$

Here the Q's represent generalized forces corresponding to the kick. Q_θ is a torque, and is related to the force Q_y by

$$Q_\theta = \frac{l}{2}Q_y. \tag{9.106}$$

. Therefore, we can eliminate the Q's from the Lagrange equations, to get

$$m\ddot{y} = \frac{2I}{l}\ddot{\theta}. \tag{9.107}$$

Integrating this once we obtain

$$m\dot{y} = \frac{2I}{l}\dot{\theta}. \tag{9.108}$$

(The constant of integration must be zero otherwise one could have a translation without a rotation.)

In the centre-of-mass frame, the plank rotates about the centre of mass. Therefore in the stationary frame, the velocity at a distance u from the centre of mass is

$$v = u\dot{\theta} + \dot{y} = \left(u + \frac{2I}{ml}\right)\dot{\theta}. \tag{9.109}$$

This will be zero when

$$u = -\frac{2I}{ml} = -\frac{2k^2}{l} = -\frac{2l^2}{12l} = -\frac{l}{6}. \tag{9.110}$$

Review

1. Introduction
   - reference frames and coordinate systems
   - vector algebra
   - vector calculus
   - 3d motion
   - Newton's laws
2. 1-d motion
   - kinetic and potential energy
   - velocity-dependent forces
3. Oscillations
   - simple pendulum
   - damped and driven harmonic oscillator
   - phase space
4. 3-d motion
   - conservative forces
   - vector operators
   - curvilinear coordinates
   - separable forces
   - 3-d harmonic oscillator
   - constrained motion
5. Noninertial reference frames
   - nonrotating frames
   - rotating frames
   - effects of Earth rotation
   - plumb bob, ballistic motion and Foucault pendulum
6. Gravitation and central forces
   - Newton's law of gravity
   - Newton's theorems
   - Kepler's laws
   - gravitational potential
   - orbital energy and velocity
   - Rutherford scattering

7. Multi-particle systems
   - centre of mass and linear momentum
   - kinetic energy and aorbital ngular momentum
   - 2-body motion
   - colisions
   - variable-mass systems
8. Rigid bodies
   - centre of mass and moment of inertia
   - kinetic energy of rotation and angular momentum
9. Lagrangian and Hamiltonian mechanics
   - Hamilton's principle
   - Lagranges equations
   - nonconservative and dissipative forces
   - conjugate momenta
   - constraints and Lagrange multipliers
   - Hamilton's equations

Final exam:

- April 27, 15:30, Hebb 100
- 2.5 hours
- closed book
- two formula sheets and simple calculator (no internet or wifi)
- expect to choose 5 of 6 questions to answer