1 Midterm Examination, 1 March 2017

This is a closed book exam. No written materials are permitted other than a formula sheet (two sided). No electronic devices are allowed other than a calculator. E-readers and internet-capable devices are prohibited.

Please answer all questions. Write your answers in the examination booklets provided. Write your name and student number on the cover. You have 50 minutes.

1. [10] A car of mass m is propelled by an engine that has a maximum power P. The engine produces a force $F = P/v$, where v is the speed of the car. The motion of the car is opposed by a frictional force that is proportional to its velocity, $F_d = -cv$, where c is a constant. If the car starts from rest and full power is applied and held, find the speed of the car as a function of time. What is the maximum speed that the car can reach?

2. [10] A mass m is suspended by a wire of length l to form a simple pendulum. The mass swings back and forth reaching a maximum angle of $\theta = \theta_0$, where θ is the angle between the wire and the vertical direction. Find the tension in the wire as a function of θ. For what value of θ is the tension greatest? Do not assume that θ is small.

Solutions are on the next page...

Answers:

1. The problem is one dimensional. The equation of motion is
$$m\frac{dv}{dt} = F = \frac{P}{v} - cv = \frac{P - cv^2}{v}.$$
This can be integrated by separation of variables,
$$m\int_0^v \frac{vdv}{P - cv^2} = \int_0^t dt = t.$$
The integral is easily done using the substitution $u = P - cv^2$, so $du = -2cvdv$. This gives
$$-\frac{m}{2c}\int_P^{P-cv^2} \frac{du}{u} = \int_0^t dt = t.$$
which has the solution
$$-\frac{m}{2c}\ln\left(\frac{P - cv^2}{P}\right) = t.$$
so
$$v = \sqrt{\frac{P}{c}(1 - e^{-2ct/m})}.$$
To find the maximum speed, let $t \to \infty$. This gives $v_{\max} = \sqrt{P/c}$.

2. Use a polar coordinate system centred at the suspension point of the wire. The position of the mass is then given by (r, θ) where $r = l$. The centripetal acceleration is
$$a = r\dot{\theta}^2 = \frac{v^2}{r}.$$
where $v = r\dot{\theta}$ is the speed of the mass. This results a centrifugal force mv^2/r that acts in the radial direction. (There is also a component of acceleration in the θ direction, due to the changing speed of the mass. But this component is perpendicular to the wire and so has no effect on the tension.) The other force acting on the mass is $m\boldsymbol{g}$ which acts downwards. Its radial component is $mg\cos\theta$. These two forces must be balanced by the tension S in the wire, so
$$S = \frac{mv^2}{r} + mg\cos\theta.$$
To find v, we can use conservation of energy. Let z be the height of the mass, above its lowest point. Then,
$$E = \frac{1}{2}mv^2 + mgz = \frac{1}{2}mv^2 + mgl(1 - \cos\theta).$$
To find E note that $v = 0$ when $\theta = \theta_0$, so
$$E = mgl(1 - \cos\theta_0).$$
Therefore,
$$v^2 = 2gl(\cos\theta - \cos\theta_0)$$
Putting this into our equation for S, we find
$$S = mg(3\cos\theta - 2\cos\theta_0).$$
This is maximum when $\theta = 0$ and is independent of the length of the pendulum.