Tutorial exercises, March 14

1. (7.11 in the text) Show that the angular momentum of a two-particle system is

$$\boldsymbol{r}_{\text{cm}} \times m\boldsymbol{v}_{\text{cm}} + \boldsymbol{R} \times \mu\boldsymbol{v}$$

where $m = m_1 + m_2$, μ is the reduced mass, $\boldsymbol{R}$ is the relative position vector, and $\boldsymbol{v}$ is the relative velocity of the two particles.

Answers:

1. The angular momentum, in an arbitrary inertial frame, is

$$\boldsymbol{L} = m_1 \boldsymbol{r_1} \times \boldsymbol{v}_1 + m_2 \boldsymbol{r_2} \times \boldsymbol{v}_2$$

The position and velocity of the centre of mass are

$$\boldsymbol{r}_{\rm cm} = \frac{1}{m}(m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2),$$
$$\boldsymbol{v}_{\rm cm} = \frac{1}{m}(m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2).$$

Also,

$$\boldsymbol{r}_2 = \boldsymbol{r}_1 - \boldsymbol{R},$$
$$\boldsymbol{v}_2 = \boldsymbol{v}_1 - \boldsymbol{v}.$$

Using these four equations, we can eliminate $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{v}_1 and \boldsymbol{v}_2$ from the equation for $\boldsymbol{L}$. First substitute the last two equations into those for the centre-of-mass quantities,

$$\boldsymbol{r}_{\rm cm} = \frac{1}{m}[m_1\boldsymbol{r}_1 + m_2(\boldsymbol{r}_1 - \boldsymbol{R}] = \boldsymbol{r}_1 - \frac{m_2}{m}\boldsymbol{R},$$
$$\boldsymbol{v}_{\rm cm} = \frac{1}{m}[m_1\boldsymbol{v}_1 + m_2(\boldsymbol{v}_1 - \boldsymbol{v})] = \boldsymbol{v}_1 - \frac{m_2}{m}\boldsymbol{v}.$$

So,

$$\boldsymbol{r}_1 = \boldsymbol{r}_{\rm cm} + \frac{m_2}{m}\boldsymbol{R},$$
$$\boldsymbol{v}_1 = \boldsymbol{v}_{\rm cm} + \frac{m_2}{m}\boldsymbol{v}.$$
$$\boldsymbol{r}_2 = \boldsymbol{r}_{\rm cm} - \frac{m_1}{m}\boldsymbol{R},$$
$$\boldsymbol{v}_2 = \boldsymbol{v}_{\rm cm} - \frac{m_1}{m}\boldsymbol{v}.$$

Putting these into the first equation, we get

$$\begin{aligned}\boldsymbol{L} &= m_1\left(\boldsymbol{r}_{\rm cm} + \frac{m_2}{m}\boldsymbol{R}\right) \times \left(\boldsymbol{v}_{\rm cm} + \frac{m_2}{m}\boldsymbol{v}\right) + m_2\left(\boldsymbol{r}_{\rm cm} - \frac{m_1}{m}\boldsymbol{R}\right) \times \left(\boldsymbol{v}_{\rm cm} - \frac{m_1}{m}\boldsymbol{v}\right), \\ &= (m_1 + m_2)\boldsymbol{r}_{\rm cm} \times \boldsymbol{v} + \left(\frac{m_1 m_2^2}{m^2} + \frac{m_1^2 m_1}{m^2}\right)\boldsymbol{R} \times \boldsymbol{v}, \\ &= m\boldsymbol{r}_{\rm cm} \times \boldsymbol{v} + \mu\boldsymbol{R} \times \boldsymbol{v},\end{aligned}$$