Tutorial exercises, March 7

1. Prove that the time average of the potential energy of a planet in an elliptical orbit about the Sun is $-k/a$, where $k = GMm$.

2. What is the time average of the kinetic energy?

Hint: the time average of a periodic function $g(t)$ is defined as

$$\langle g \rangle = \frac{1}{T} \int_0^T g(t) dt.$$

The following integral may be useful

$$\int_0^{2\pi} \frac{d\theta}{1 + \epsilon \cos\theta} = \frac{2\pi}{\sqrt{1 - \epsilon^2}}.$$

Answers:

1. The potential energy of the planet is given by

$$V = -\frac{k}{r}.$$

The time average will be

$$\langle V \rangle = -\frac{k}{T}\int_0^T \frac{dt}{r(t)} = -\frac{k}{T}\int_0^{2\pi} \frac{dt}{d\theta}\frac{d\theta}{r} = -\frac{k}{T}\int_0^{2\pi} \frac{d\theta}{r\dot{\theta}}$$

Now, recall that the angular momentum per unit mass is $l = r^2\dot{\theta}$, so this can be simplified,

$$\langle V \rangle = -\frac{k}{lT}\int_0^{2\pi} r d\theta.$$

Now substitute $r = a(1-\epsilon^2)/(1+\epsilon\cos\theta)$ from the equation of an ellipse,

$$\langle V \rangle = -\frac{ka(1-\epsilon^2)}{lT}\int_0^{2\pi} \frac{d\theta}{1+\epsilon\cos\theta}.$$

Now substitute $l^2 = GMa(1-\epsilon^2)$ and $T^2 = 4\pi^2a^3/GM$,

$$\langle V \rangle = -\frac{ka(1-\epsilon^2)}{2\pi a^2\sqrt{1-\epsilon^2}}\int_0^{2\pi} \frac{d\theta}{1+\epsilon\cos\theta} = -\frac{GMm}{a}.$$

2. The total energy is given by

$$E = T + V = -\frac{GMm}{2a}$$

(here we are using T to represent kinetic energy). Therefore

$$\langle T \rangle = \langle E - V \rangle = \langle E \rangle - \langle V \rangle = E + \frac{GMm}{a} = \frac{GMm}{2a}.$$