Tutorial exercises, February 28

1. Prove that if a forced is central (has a radial component only), and isotropic (does not depend on θ or ϕ), then it is conservative.
2. Assuming a circular orbit, show that Kepler's third law follows directly from his second law and his law of gravity: $GMm/r^2 = mv^2/r$.

Answers:

1. Any central force can be written in the form

$$\boldsymbol{F} = f(r, \theta, \phi)\boldsymbol{e}_r.$$

To prove that it is conservative, it is sufficient to show that the curl of the force is zero. In spherical coordinates the curl is (see Section 4.2.6 in the notes)

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \boldsymbol{e}_r & r\boldsymbol{e}_\theta & r\sin\theta\boldsymbol{e}_\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix},$$

The only nonzero component of the force is the radial component, therefore,

$$\boldsymbol{\nabla} \times \boldsymbol{F} = \frac{1}{r^2 \sin\theta} \left(r\boldsymbol{e}_\phi \frac{\partial F_r}{\partial \phi} - r\sin\theta\boldsymbol{e}_\phi \frac{\partial F_r}{\partial \theta} \right),$$

Now, if the force is also isotropic, $F_r = f(r)$ depends only on r, so derivatives with respect to θ and ϕ are zero. Therefore

$$\boldsymbol{\nabla} \times \boldsymbol{F} = 0,$$

2. For a circular orbit, $r = a$. The velocity can be found by dividing the circumference of the orbit by the period,

$$v = \frac{2\pi a}{T}$$

Therefore,

$$\frac{GMm}{a^2} = \frac{mv^2}{a} = \frac{4\pi^2 m a^2}{T^2 a}$$

so

$$T^2 = \frac{4\pi^2 a^3}{GM}.$$