Tutorial exercises, Feburary 7

1. (This is Question 4.6 in the text book)

   Consider the two force functions

$$\boldsymbol{F} = x\boldsymbol{i} + y\boldsymbol{j} \tag{1}$$
$$\boldsymbol{F} = y\boldsymbol{i} - x\boldsymbol{j} \tag{2}$$

   Verify that (1) is conservative and that (2) is nonconservative by showing that $\int \boldsymbol{F} \cdot d\boldsymbol{r}$ is independent of the path of integration for (1) but not for (2), by taking two paths in which the starting point is the origin (0,0), and the endpoint is (1,1). For one path thake the line $y = x$. For the other path take the x-axis out to the point (1,0) and then the line $x = 1$ up to the point (1,1).

Answers:

(A picture showing the paths should be drawn)

1. The dot product can be written in terms of components,

$$\boldsymbol{F} \cdot d\boldsymbol{r} = F_x dx + F_y dy.$$

For the force of Equation (1) this becomes

$$\boldsymbol{F} \cdot d\boldsymbol{r} = xdx + ydy.$$

Let A be the point (0,0), B be (1,1), and C be (1,0). Along the first path, $y = x$, so $dy = dx$. The line integral is

$$\int_A^B \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^1 (xdx + ydy) = 2\int_0^1 xdx = 1.$$

Along the second path the integral is separated into two parts,

$$\int_A^B \boldsymbol{F} \cdot d\boldsymbol{r} = \int_A^C \boldsymbol{F} \cdot d\boldsymbol{r} + \int_C^B \boldsymbol{F} \cdot d\boldsymbol{r}.$$

Along the path from A to C, $y = dy = 0$, and along the path from C to B, $x = 1, dx = 0$. Therefore,

$$\int_A^B \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^1 xdx + \int_0^1 ydy = 1.$$

As expected, these give the same result.

For the force of Equation (2) we have

$$\boldsymbol{F} \cdot d\boldsymbol{r} = ydx - xdy.$$

The line integral along the first path is

$$\int_A^B \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^1 (ydx - xdy) = \int_0^1 (xdx - xdx) = 0.$$

Along the second path we get,

$$\int_A^B \boldsymbol{F} \cdot d\boldsymbol{r} = \int_A^C (ydx - xdy) + \int_C^B (ydx - xdy) = 0 + \int_0^1 (0 - dy) = 1.$$

In this case the results differ, so the force is not conservative.