

Tutorial exercises, January 31

1. Prove the following vector identities:

$$\begin{aligned}\nabla \times \nabla f &= 0, \\ \nabla \cdot (\nabla \times \mathbf{A}) &= 0.\end{aligned}$$

Answers:

1.

$$\begin{aligned}\nabla \times \nabla f &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \end{vmatrix} \\ &= \left[\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \right] \mathbf{i} + \left[\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \right] \mathbf{j} + \left[\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right] \mathbf{k} = 0.\end{aligned}$$

Each term vanishes because the order of partial differentiation does not matter. Similarly,

$$\begin{aligned}\nabla \cdot (\nabla \times \mathbf{A}) &= (\mathbf{i}\partial/\partial x + \mathbf{j}\partial/\partial y + \mathbf{k}\partial/\partial z) \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}, \\ &= \frac{\partial^2 A_z}{\partial x \partial y} - \frac{\partial^2 A_y}{\partial x \partial z} + \frac{\partial^2 A_x}{\partial y \partial z} - \frac{\partial^2 A_z}{\partial y \partial x} + \frac{\partial^2 A_y}{\partial z \partial x} - \frac{\partial^2 A_x}{\partial z \partial y} = 0.\end{aligned}$$