Tutorial exercises, January 24

1. Let a particle of unit mass be subject to a force $x - x^3$, where x is its displacement from the coordinate origin.
   (a) Find the equilibrium points (points at which a particle with zero velocity will remain at rest), and determine whether they are stable or unstable.
   (b) Calculate the period with which the particle will oscillate about a stable equilibrium point, assuming that the amplitude of the oscillation is small.
   (c) Sketch the trajectories of the particle in phase space.

Answers:

1. (a) The force on the particle will be zero when $x - x^3 = 0$, which has the solutions $x = \pm 1$ and $x = 0$. These are the two equilibrium points, since if the particle is placed at either, with zero velocity, it will remain there. To see if they are stable, we need to look at the potential.
   The potential corresponding to this force is
   $$V(x) = -\int F dx = -\frac{1}{2}x^2 + \frac{1}{4}x^4.$$
   To determine the stability, we compute the second derivative of V and evaluate it at the equilibrium points.
   $$\frac{d^2V}{dx^2} = \frac{d}{dx}(-x + x^3) = -1 + 3x^2$$
   This is negative for $x = 0$, so this point is unstable. For $x = \pm 1$, the second derivative is positive, so these two points are stable.
   (b) Consider the motion of a particle near the stable equilibrium point $x = 1$. Let $u = x - 1$ be the displacement of the particle from this point. In terms of u, the force on the particle will be
   $$F = (1 + u) - (1 + u)^3 = 1 + u - 1 - 3u + 3u^2 - u^4 = -2u + 3u^2 + 3u^3.$$
   For small displacements, we can keep only the linear term and ignore the higher-order terms. Therefore, the equation of motion is
   $$\ddot{u} = -2u = -ku.$$
   This is a harmonic oscillator with angular frequency $\omega_0 = \sqrt{k/m} = \sqrt{2}$. The period is
   $$T = \frac{2\pi}{\omega_0} = \sqrt{2}\pi$$
   An equivalent way to determine the frequency is to remember that the spring constant k is related to the second derivative of the potential (or equivalently the first derivative of the force), evaluated at the equilibrium point
   $$k = \left[\frac{d^2V(x)}{dx^2}\right]_{x_0} = -\left[\frac{dF(x)}{dx}\right]_{x_0} = 2$$

(c) Start by drawing a sketch of the potential V(x). It is zero at $x = 0$ and curves downward reaching a minimum when $|x| = 1$. It then rises without limit when $|x| > 1$. The motion is therefore bounded. If $E > 0$, the particle will oscillate back and forth between turning points at $x = \pm x_t$ defined by $V(x_t) = E$. If $E < 0$ the particle will be trapped in one of the two potential wells and will oscillate about $x = 1$ or $x = -1$. The phase-space trajectories will be closed loops surrounding $x = \pm 1$ surrounded by a family of larger loops (distorted ellipses) that are symmetric about the x and v axes.