Tutorial exercises, April 4

1. A solid uniform sphere of radius a has a spherical cavity of radius $a/2$ centred at a point $a/2$ from the centre of the sphere.
   (a) Find the centre of mass.
   (b) Find the moment of inertia, and the radius of gyration, about an axis passing through the centre of the sphere and the centre of the cavity.

Answers:

1. (a) By symmetry, the centre of mass must lie along the axis passing through the centre of the sphere and the centre of the cavity. Choose this to be the z axis and chose the centre of the sphere as the origin of the Cartesian coordinate system. Then, the distance to the centre of mass along this axis will be

$$z_{\mathrm{cm}} = \frac{1}{m}\int_V z\rho dV,$$

where m is the mass. Let V_1 be a solid sphere of radius a and V_2 be a solid sphere of radius $a/2$, displaced from the origin by a distance $a/2$.
The mass is just the mass of V_1 minus the mass of V_2,

$$m = \frac{4\pi\rho}{3}a^3 - \frac{4\pi\rho}{3}\left(\frac{a}{2}\right)^3 = \frac{7}{8}\frac{4\pi\rho}{3}a^3 = \frac{7\pi\rho}{6}a^3.$$

The integral can be written as the difference between the integral over V_1 and the integral over V_2,

$$z_{\mathrm{cm}} = \frac{\rho}{m}\left[\int_{V_1} zdV - \int_{V_2} zdV\right].$$

The first integral is zero since x is antisymmetric about the origin. The second integral is the volume of V_2 times the mean distance, $a/2$ from the origin. Therefore,

$$z_{\mathrm{cm}} = -\frac{\rho a}{2m}\int_{V_2} dV = -\frac{(a/2)(1/8)}{7/8} = -\frac{a}{14}.$$

So the centre of mass is in the direction opposite the void, a distance $a/14$ from the centre of the sphere.

(b) In the same manner, we can write the moment of inertia as

$$I = \rho\left[\int_{V_1}(x^2+y^2)dV - \int_{V_2}(x^2+y^2)dV\right].$$

This is the moment of inertia of a sphere of radius a minus the moment of inertia of a sphere of radius $a/2$ (the position along the axis of rotation does not matter). Since the moment of inertia of a uniform sphere of mass M and radius r is $2Mr^2/5$, we have

$$I = \frac{4\pi\rho a^3}{3}\frac{2}{5}\left[a^2 - \frac{1}{8}\left(\frac{a}{2}\right)^2\right] = \frac{4\pi\rho a^3}{3}\frac{2}{5}\frac{31}{32}a^2 = \frac{31\pi\rho a^5}{60}.$$

The radius of gyration is k where

$$k^2 = \frac{I}{m} = \frac{31\pi\rho a^5/60}{7\pi\rho a^3/6} = \frac{31}{70}a^2$$