Tutorial exercises, March 28

1. Find the acceleration of a solid uniform sphere rolling without slipping down a fixed inclined plane of angle θ (measured from the horizontal direction).

Answers:

1. We need only one parameter to specify the position of the sphere, so the problem is one-dimensional. Let $q_1 = x$, the distance that the centre of the sphere has moved along the plane from its starting point. Let a be the radius of the sphere.

   Because there is no slipping, the angular velocity of rotation of the sphere is related to its velocity by
   $$\omega = \frac{\dot{x}}{a}.$$
   The kinetic energy is the sum of the translational energy of the centre of mass and the rotational energy,
   $$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\frac{I}{a^2}\dot{x}^2.$$
   The potential energy is mg times the height of the centre of mass,
   $$V = -mgx\sin\theta$$
   Therefore, the Lagrangian is
   $$L = \frac{1}{2}\left(m + \frac{I}{a^2}\right)\dot{x}^2 + mgx\sin\theta.$$
   There is one Lagrange equation,
   $$\frac{d}{dt}\left[\left(m + \frac{I}{x}\right)\dot{x}\right] - mg\sin\theta.$$
   which gives the equation of motion
   $$\ddot{x} = \frac{mg\sin\theta}{m + I/a^2} = \frac{g\sin\theta}{1 + k^2/a^2},$$
   where $k^2 = I/m$. For a sphere, $k^2 = 2a^2/5$, so the final result is
   $$\ddot{x} = \frac{g\sin\theta}{1 + 2/5} = \frac{5}{7}g\sin\theta.$$