1 Tutorial excercises, January 10

1. (F&C 1.4) Consider a cube whose edges are each of unit length. One corner coincides with the origin of an xyz Cartesian coordinate system. Three of the cube's edges extend from the origin along the positive direction of each coordinate axis. Find the vector that begins at the origin and extends
   (a) along a major diagonal of the cube;
   (b) along the diagonal of the lower face of the cube.
   (c) Calling these vectors $\boldsymbol{A}$ and $\boldsymbol{B}$, find $\boldsymbol{C} = \boldsymbol{A} \times \boldsymbol{B}$.
   (d) Find the angle between $\boldsymbol{A}$ and $\boldsymbol{B}$.
2. (F&C 1.14) Express the vector $2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k}$ in terms of the unit vectors $\boldsymbol{i}'\boldsymbol{j}'\boldsymbol{k}'$ in which the x' and y' axes are rotated about the z axis through an angle of 30°.
3. (F&C 1.30) Use vector algebra to derive the following trigonometric identities
   (a) $\cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$
   (b) $\sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$
4. We have seen that rotations can be represented by orthogonal matrices. Prove the following statements about orthogonal matrices (of equal rank)
   (a) the product of two orthogonal matrices is an orthogonal matrix
   (b) every orthogonal matrix has an inverse
   (c) there is an orthogonal matrix (the identity matrix) that leaves others unchanged

   These are the necessary and sufficient conditions for orthogonal matrices to form a group, in the mathematical sense, called O(3) in three dimensions.

Answers:

1. (a) (1,1,1)
   (b) (1,1,0)
   (c) (-1,1,0)
   (d) $\boldsymbol{A} \cdot \boldsymbol{B} = 2 = AB\cos\theta = \sqrt{3}\sqrt{2}\cos\theta$ so $\theta = \arccos(\sqrt{2/3}) = 35.26°$.
2. $$\boldsymbol{x}' = \begin{pmatrix} \sqrt{3}/2 & 1/2 & 0 \\ -1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} \sqrt{3} + 3/2 \\ 3\sqrt{3}/2 - 1 \\ -1 \end{pmatrix}$$

   so $2\boldsymbol{i} + 3\boldsymbol{j} - \boldsymbol{k} = (\sqrt{3} + 3/2)\boldsymbol{i}' + (3\sqrt{3}/2 - 1)\boldsymbol{j}' - \boldsymbol{k}'$.
3. Define unit vectors $\boldsymbol{A} = \cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}$ and $\boldsymbol{B} = \cos\phi\boldsymbol{i} + \sin\phi\boldsymbol{j}$. Now compute $\boldsymbol{A} \cdot \boldsymbol{B}$ and $\boldsymbol{A} \times \boldsymbol{B}$ (draw figure showing these vectors to help students visualize).

4. (a) Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be orthogonal matrices and $\boldsymbol{C} = \boldsymbol{B}\boldsymbol{A}$. Then

$$\boldsymbol{C}^T\boldsymbol{C} = (\boldsymbol{B}\boldsymbol{A})^T(\boldsymbol{B}\boldsymbol{A}) = \boldsymbol{A}^T\boldsymbol{B}^T\boldsymbol{B}\boldsymbol{A} = \boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I},$$

which proves that $\boldsymbol{C}$ is orthogonal.

(b) Every matrix has a transpose, so every orthogonal matrix has an inverse.

(c) The unit matrix $\boldsymbol{I}$ is its own inverse and is equal to its transpose, so it is orthogonal.