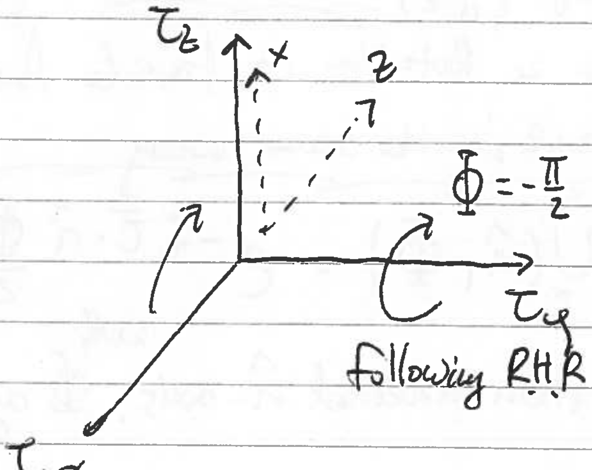
Supplementary Materials on other Representations

Representations for Dirac Algebras are not unique. α-β Representation is the one originally introduced but γ-matrices are widely used in particle physics / Q.F.T. γ-matrices can be obtained by Rotations / unitary transformation.

Take massless limit for Example.

$$H_D = \tau_x \otimes \vec{\sigma}\cdot\vec{p} \xrightarrow[\text{around } \tau_y]{\text{Rotation}} \underbrace{H_D' = \tau_z \otimes \vec{\sigma}\cdot\vec{p}}_{\text{yields Weyl fermions}}$$



SU(2)

define a Rotation in Particle-Anti Particle Space in the same way

$$U_\tau(\hat{n}, \Phi) = e^{-i\vec{\tau}\cdot\hat{n}\frac{\Phi}{2}}.$$

with (Rotation around $\hat{n}$-axis, Φ angle)

To have $H_D \to H_D'$, we chose $\hat{n} = \hat{e}_y$, $\Phi = -\frac{\pi}{2}$

$$\psi \to U_\tau(\hat{e}_y, -\tfrac{\pi}{2})\psi, \quad H_D \to H_D' = U_\tau(\hat{e}_y, -\tfrac{\pi}{2}) H_D U_\tau^\dagger(\hat{e}_y, -\tfrac{\pi}{2})$$

$$= U_\tau^\dagger(\hat{e}_y, \tfrac{\pi}{2}) H_D U_\tau(\hat{e}_y, \tfrac{\pi}{2})$$

$$H_D' = \frac{1}{\sqrt{2}}(1 + i\tau_y)\, \tau_x \otimes \vec{\sigma}\cdot\vec{p}\, \frac{1}{\sqrt{2}}(1 - i\tau_y)$$

$$= \tau_z \otimes \vec{\sigma}\cdot\vec{p}.$$

by the same token, $m\beta \to -\tau_x m$ "τ_z"

In Rotated basis, $[i\partial_t - H_D'] \psi' = 0$,

$$H_D' = \begin{bmatrix} \vec{\sigma}\cdot\vec{p}, & 0 \\ 0, & -\vec{\sigma}\cdot\vec{p} \end{bmatrix} + \begin{bmatrix} 0 & -m \\ -m & 0 \end{bmatrix}$$

To cast into γ matrices, you need to further multiply the Dirac Equation with a matrix so that the mass is coupled to $\mathbb{1}_{4\times4}$ <u>4×4 unity matrix</u>.

Projection Operator for Weyl fermions are (in α-β Rep)

$$P_\pm = \frac{1 + \underbrace{\tau_x \otimes \mathbb{1}}_{\gamma_5}}{2}, \qquad [P_\pm, H_D]_{m=0} = 0$$

<u>commuting with H_D</u> $m=0$

$$i\partial_t \psi = H_0 \psi, \qquad \psi = P_+\psi + P_-\psi$$
$$= \psi_R + \psi_L$$

Applying $P_\pm \psi$ To Dirac Equation taking into account $m=0$, One has

$$i\partial_t \psi_R = H_D \psi_R, \quad i\partial_t \psi_L = H_D \psi_L$$

$$H_D \psi_R = \mathbb{1}\otimes\vec{\sigma}\cdot\vec{p}\ \psi_R, \qquad H_D \psi_L = -\mathbb{1}\otimes\vec{\sigma}\cdot\vec{p}$$

effective two component

$$\psi_R = \begin{bmatrix} \chi_R \\ \chi_R \end{bmatrix} \frac{1}{\sqrt{2}} \qquad \psi_L = \begin{bmatrix} \chi_L \\ -\chi_L \end{bmatrix} \frac{1}{\sqrt{2}}$$