Supplementary Materials on $\hat{T}$ matrix

Formally speaking

$$|\psi_{TOT,\vec{k}_i}\rangle = |\vec{k}_i\rangle + \underbrace{\frac{1}{E-H_0} V |\psi_{TOT,\vec{k}_i}\rangle}_{|\psi_{SC},\vec{k}_i\rangle}$$

Method of iteration leads to

$$|\psi_{TOT,\vec{k}_i}\rangle = |\vec{k}_i\rangle + \frac{1}{E-H_0} V |\vec{k}_i\rangle + \frac{1}{E-H_0} V \frac{1}{E-H_0} V |\vec{k}_i\rangle$$

$$+ \cdots\cdots$$

Real space wave function

$$\psi_{TOT,\vec{k}_i}(\vec{x}) = \langle \vec{x} | \psi_{TOT,\vec{k}_i}\rangle$$

$$= \frac{1}{\sqrt{\Omega}} e^{i\vec{k}_i\cdot\vec{x}} + \int_{\vec{x}'} G_E(\vec{x},\vec{x}') \langle \vec{x}' | V | \psi_{TOT,\vec{k}_i}\rangle$$

$$\xrightarrow{|\vec{x}|\to\infty} \frac{1}{\sqrt{\Omega}} \left(e^{i\vec{k}_i\cdot\vec{x}} + \frac{e^{ik|\vec{x}|}}{|\vec{x}|} \left[-\frac{2}{4\pi} T_{\vec{k}_f,\vec{k}_i}(E) \right] \right)$$

Hence $\boxed{f(\theta) = -\frac{2}{4\pi} T_{\vec{k}_f,\vec{k}_i}(E)}$ General Stuff

$$T_{\vec{k}_f,\vec{k}_i}(E) = \Omega \langle \vec{k}_f | V | \psi_{TOT,\vec{k}_i}\rangle$$

Note that $\hat{k}_f \cdot \hat{k}_i = \cos\theta$,

$$G_E(\vec{x},\vec{x}') = -\frac{2}{4\pi} \frac{e^{ik|\vec{x}-\vec{x}'|}}{|\vec{x}-\vec{x}'|}, \quad \frac{k^2}{2} = E$$

One can show easily that

$$T_{\vec{k}_f,\vec{k}_i}(E) = V_{\vec{k}_f,\vec{k}_i} + \frac{1}{\Omega}\sum_{k'} V_{\vec{k}_f,\vec{k}'} \times \frac{1}{E-\frac{k'^2}{2}} T_{\vec{k}',\vec{k}}(E)$$

where $V_{\vec{k}_f,\vec{k}_i} = \Omega \langle \vec{k}_f | v | \vec{k}_i \rangle$

or

$$\boxed{T_{\vec{k}_f,\vec{k}_i}(E) = V_{\vec{k}_f,\vec{k}_i} + \int \frac{d^3\vec{k}'}{(2\pi)^3} V_{\vec{k}_f,\vec{k}'}\, G_E(\vec{k}')\, T_{\vec{k}',\vec{k}}(E)}$$

$V_{\vec{k}_f,\vec{k}_i} = \langle \vec{k}_f | v | \vec{k}_i \rangle$ if you set $\Omega = 1$ everywhere

$G_E(\vec{k}') = \langle k' | \frac{1}{E-H_0+i\delta} | k' \rangle \Leftarrow$ Green function

($\delta > 0$ infinetesmal positive)

$$= \boxed{\frac{1}{E-\frac{k'^2}{2}+i\delta}}$$

Via iteration, one has

$$T_{\vec{k}_f,\vec{k}_i}(E) = V_{\vec{k}_f,\vec{k}_i} + \int V_{\vec{k}_f,\vec{k}'}\, G_E(k')\, V_{\vec{k}',\vec{k}_i} +$$

$$+ \int\int V_{\vec{k}_f,\vec{k}'}\, G_E(k')\, V_{\vec{k}',k''}\, G_E(k'')\, V_{\vec{k}'',\vec{k}_i} + \cdots$$

in form of $O(V^0)$, $O(V^1)$, $O(V^2)$, $O(V^3)$ . . .

Finally,

$$f(\theta) = \sum_\ell [4\pi(2\ell+1)]^{1/2} Y_{\ell,0}(\theta) \overbrace{\frac{\sin\delta_\ell e^{i\delta_\ell}}{k}}^{a_\ell(k)}$$

$$\frac{\sin\delta_\ell e^{i\delta_\ell}}{k} = \frac{1}{[4\pi(2\ell+1)]^{1/2}} \int Y_\ell^*(\theta) f(\theta)\, d\Omega$$

Solid angle 4π

$$= -\frac{2}{4\pi} \frac{1}{[4\pi(2\ell+1)]^{1/2}} \int_{S.A.} Y_\ell(\theta) T_{\vec{k}_f, \vec{k}_i}(E)\, d\Omega$$

$d\Omega = \sin\theta\, d\theta\, d\varphi$, $\quad \hat{k}_f \cdot \hat{k}_i = \cos\theta$

S-wave limit, $\ell = 0$, $\quad Y_0(\theta) = \frac{1}{(4\pi)^{1/2}}$

$$\frac{\sin\delta_0 e^{i\delta_0}}{k} = -\frac{2}{4\pi} \langle T_{\vec{k}_f, \vec{k}_i}(E) \rangle$$

$\langle \cdots \rangle$ Average over (θ, φ) $= \frac{1}{4\pi} \int d\Omega \ldots$