Supplementary Stuff on Resonance Conditions etc

A) For square well potential, we have

$$\tan\delta_\ell = \frac{j_\ell'(\rho)}{n_\ell'(\rho)} \left. \frac{\left[1 - \tilde{\rho}\frac{j_\ell'(\tilde{\rho})}{j_\ell(\tilde{\rho})} \Big/ \rho\frac{j_\ell'(\rho)}{j_\ell(\rho)}\right]}{\left[1 - \tilde{\rho}\frac{j_\ell'(\tilde{\rho})}{j_\ell(\tilde{\rho})} \Big/ \rho\frac{n_\ell'(\rho)}{n_\ell(\rho)}\right]} \right|_{\substack{\rho=\rho_0 \\ \tilde{\rho}=\tilde{\rho}_0}}$$

$\rho = kr$, $\tilde{\rho}^2 = 2B + \rho^2$. $B = V_0 r_0^2$ – Born parameter

Note that $\rho_0 \ll 1$ but $B \sim O(1)$ near Resonance.

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$$\frac{j_\ell'(\rho)}{n_\ell'(\rho)} \overset{\rho\to 0}{=} + \frac{\ell}{\ell+1} \times \frac{1}{(2\ell+1)!!} \times \frac{1}{(2\ell-1)!!}\, \rho^{2\ell+1} \qquad \text{(See HW set 0)}$$

This leads relatively small phase shifts in $\ell \neq 0$ channel.

* As

$$\lim_{\substack{\rho\to 0\\ \tilde{\rho}\to 0}} \tilde{\rho}\frac{j_\ell'(\tilde{\rho})}{j_\ell(\tilde{\rho})} \Big/ \rho\frac{j_\ell'(\rho)}{j_\ell(\rho)} = 1, \qquad 1 - \frac{\tilde{\rho}\frac{j_\ell'(\tilde{\rho})}{j_\ell(\tilde{\rho})}}{\rho\frac{j_\ell'(\rho)}{j_\ell(\rho)}} \overset{B\to 0}{=} o(B)$$

* Resonance Condition (for $k \to 0$)

$$\boxed{1 - \tilde{\rho}\frac{j_\ell'(\tilde{\rho})}{j_\ell(\tilde{\rho})} \Big/ \rho\frac{n_\ell'(\rho)}{n_\ell(\rho)} = 0 \quad \text{when } \rho = \rho_0 \to 0}$$

As $\rho \frac{n_\ell'(\rho)}{n_\ell(\rho)} \overset{\rho\to 0}{=} -(\ell+1)$, this leads to, for any "ℓ",

$$\tilde{\rho}_0 j_\ell'(\tilde{\rho}_0) + (\ell+1) j_\ell(\tilde{\rho}_0) = 0 \qquad \text{or} \qquad \frac{d}{d\tilde{\rho}}\left[\tilde{\rho}_0^{\,\ell+1} j_\ell(\tilde{\rho}_0)\right] = 0$$

$\left(\tilde{\rho}_0 = \sqrt{2V_0 r_0^2} = \sqrt{2B}\right)$. The solution yield B_ℓ^*, $\ell = 0, 1, 2 \ldots$

B)

The bound state solution $B_\ell(\rho_B) = h_\ell^{(1)}(i k_B r)$

$\rho_B = k_B r$, $E_b = \frac{k_B^2}{2} = -E$, $B_\ell(\rho_B) \overset{\rho_B\to\infty}{=} \frac{e^{-\rho_B}}{\rho_B}$

Following the asymptotic property of $h^{(1)}(\rho)$. (see HW set 0)

$$\begin{cases} B_\ell(\rho_B) = \rho_B\left[-\frac{1}{\rho_B}\frac{\partial}{\partial \rho_B}\right]^\ell B_0(\rho_B) & \text{identical to } h_\ell^{(1)}(\rho). \\ B_0(\rho_B) = \frac{e^{-\rho_B}}{\rho_B}, & B_\ell(\rho_B) \xrightarrow{\rho_B\to 0} \frac{1}{\rho_B^{\,\ell+1}}(2\ell-1)!! \end{cases}$$

To have a bound state, One requires that

$$\left.\frac{\tilde{\rho}\, j_\ell'(\tilde{\rho})}{j_\ell(\tilde{\rho})}\right|_{\tilde{\rho}=\tilde{\rho}_0} = \left.\rho_B \frac{B_\ell'(\rho_B)}{B_\ell(\rho_B)}\right|_{\rho_B=\rho_{B_0}}, \quad \begin{cases} \rho_{B_0} = k_B r_0 \\ \tilde{\rho}^2 = 2B - \rho_B^2 \\ \tilde{\rho}_0^2 = 2B - \rho_{B_0}^2 \end{cases}$$

when $k_B = E_b = 0$, or at the critical value of B^*.

$$\left.\tilde{\rho}\,\frac{j_\ell'(\tilde{\rho})}{j_\ell(\tilde{\rho})}\right|_{\tilde{\rho}=\sqrt{2B}} = \left.\rho_B\,\frac{B_\ell'(\rho_B)}{B_\ell(\rho_B)}\right|_{\rho_B\to 0} = -(\ell+1)$$

or

$$\left.\tilde{\rho}\,j_\ell'(\tilde{\rho}) + (\ell+1)\,j_\ell(\tilde{\rho})\right|_{\tilde{\rho}=\sqrt{2B^*}} = 0$$

This fixed the value of B^* at which bound state first appears and is identical to Resonance Condition above.

To understand $E_b = E_b(B-B^*)$, we expand the equation LHS near $E_b=0$, RHS near B^*.

$$\begin{cases} \left.\tilde{\rho}\,\dfrac{j_\ell'(\tilde{\rho})}{j_\ell(\tilde{\rho})}\right|_{\tilde{\rho}^2 = B-\rho_{B_0}^2} \simeq \overset{\text{Value at } B=B^*,\ \rho_{B_0}=0}{-(\ell+1)} + A_\ell\,\underbrace{(B-\rho_{B_0}^2-B^*)}_{(B-B^*)-\rho_{B_0}^2} + \cdots \\[2ex] \left.\rho_B\,\dfrac{B_\ell'(\rho_B)}{B_\ell(\rho_B)}\right|_{\tilde{\rho}_B=\rho_{B_0}} = -(\ell+1) + C_\ell\,\rho_{B_0} + D_\ell\,\rho_{B_0}^2 + \cdots \end{cases}$$

[C_ℓ is zero for $\ell\neq 0$ As mentioned during my lecture]

A_ℓ, D_ℓ, C_ℓ are $O(1)$ numbers which can be computed explicitly for Given ℓ although in the HW set II you can assume known to you. but show when $C_\ell = 0$.