Relation between phase shifts δ_ℓ and $a_B \nearrow \infty$, bound state.

Following our discussion this Monday, we find

* For phase shift

$$\frac{1 - \tan\delta_\ell(B,\rho_0)\dfrac{n_\ell'(\rho_0)}{j_\ell'(\rho_0)}}{1 - \tan\delta_\ell(B,\rho_0)\dfrac{n_\ell(\rho_0)}{j_\ell(\rho_0)}} = \frac{\tilde{\rho}_0 \dfrac{j_\ell'(\tilde{\rho}_0)}{j_\ell(\tilde{\rho}_0)}}{\rho_0 \dfrac{j_\ell'(\rho_0)}{j_\ell(\rho_0)}}$$

or

$$\tan\delta_\ell(B,\rho_0) = \frac{j_\ell'(\rho_0)}{n_\ell'(\rho_0)} \underbrace{\left[\frac{1 - \dfrac{\tilde{\rho}_0}{\rho_0}\dfrac{j_\ell'(\tilde{\rho}_0)}{j_\ell(\tilde{\rho}_0)}\dfrac{j_\ell(\rho_0)}{j_\ell'(\rho_0)}}{1 - \dfrac{\tilde{\rho}_0}{\rho_0}\dfrac{j_\ell'(\tilde{\rho}_0)}{j_\ell(\tilde{\rho}_0)}\cdot\dfrac{n_\ell(\rho_0)}{n_\ell'(\rho_0)}}\right]}_{\sim \tilde{V}_\ell(B,\rho_0)}$$

Note that

$$\tilde{\rho}_0 = \sqrt{B+\rho_0^2}, \quad \rho_0 = kR_0$$

$$\tilde{\tilde{\rho}}_0 = \sqrt{B} = \tilde{\rho}_0(\rho_0 = 0)$$

*

And for bound state, $E = -E_B = -\dfrac{1}{2a_B^2} = -\dfrac{k_B^2}{2}$

$$\rho_0\frac{\bar{h}_\ell'(\rho_0)}{\bar{h}_\ell(\rho_0)} = \frac{\tilde{\rho}_0\, j_\ell'(\tilde{\rho}_0)}{j_\ell(\tilde{\rho}_0)},$$

$\rho_0 = k_B R_0, \quad \tilde{\rho}_0 = \sqrt{B-\rho_0^2}$

$$\begin{cases} \bar{h}_\ell(\rho_0) = \rho^\ell\left[-\dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\right]^\ell \bar{h}_0(\rho_0), \\ \bar{h}_0(\rho_0) = \dfrac{e^{-\rho_0}}{\rho_0} \end{cases}$$

for bound states

$\rho_0 \to 0$ implies that

$$\tilde{\tilde{\rho}}_0\, j_\ell'(\tilde{\tilde{\rho}}_0) + (\ell+1)\, j_\ell(\tilde{\tilde{\rho}}_0) = 0.$$

under this condition, One also shows that

$$\tilde{V}_\ell(B,\rho_0=0) = +\infty \quad \text{as} \quad \frac{\rho_0 n_\ell'(\rho_0)}{n_\ell} = -[\ell+1]$$