Supplementary Materials on 3D spherical waves

3D equation $\quad -\frac{\nabla^2}{2}\psi = E\psi, \quad k^2 = 2E$

After separation of variables, one can rewrite

in Spherical Coordinates, $\quad \psi = R_\ell(kr)\, Y_{\ell,m}(\theta,\varphi)$

$$\begin{cases} -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}R_\ell\right) + \frac{\ell(\ell+1)}{r^2}R_\ell = k^2 R_\ell \\ L^2\, Y_{\ell,m}(\theta,\varphi) = \ell(\ell+1)\, Y_{\ell,m}(\theta,\varphi), \quad m = 0, \pm 1, \cdots \pm \ell \end{cases}$$

$$L^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\,\cdot\right) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

angular Momomentum operator

$$R_\ell(kr) = j_\ell(kr) \text{ or } n_\ell(kr)$$

* $\ell = 0$, $\quad \underbrace{j_0(\rho) = \frac{\sin\rho}{\rho} \qquad n_0(\rho) = -\frac{\cos\rho}{\rho}}_{\text{S-wave Bessel function and Newmann function}}, \quad \rho = kr$

One can verify

One can have

* $\ell \neq 0$,

$$\begin{cases} j_\ell(\rho) = \rho^\ell\left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}\right]^\ell j_0(\rho) \\ n_\ell(\rho) = \rho^\ell\left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}\right]^\ell n_0(\rho) \end{cases}$$

(This will be discussed in next lecture)

Given $j_0(\rho)$, $n_0(\rho)$, and $j_\ell(\rho)$, $n_\ell(\rho)$ generated via the above relations, one can show that

Eq. 1

$$j_\ell(\rho) \xrightarrow{\rho\to 0} \frac{\rho^\ell}{(2\ell+1)!!}\,, \qquad n_\ell(\rho) \xrightarrow{\rho\to 0} -\frac{(2\ell-1)!!}{\rho^{\ell+1}}$$

short distance asymptotics

Eq. 2.

$$j_\ell(\rho) \xrightarrow{\rho\to\infty} \frac{\sin(\rho - \frac{\ell\pi}{2})}{\rho}\,, \qquad n_\ell(\rho) \xrightarrow{\rho\to\infty} -\frac{1}{\rho}\cos(\rho - \tfrac{\ell\pi}{2})$$

large distance asymptotics

Hankel functions $\underbrace{h^{(1)}_\ell(\rho)}_{\text{out going}}$, $\underbrace{h^{(2)}_\ell(\rho)}_{\text{incoming}}$

$$h^{(1)}_\ell(\rho) = j_\ell(\rho) + i\,n_\ell(\rho)\,, \qquad h^{(2)}_\ell(\rho) = j_\ell(\rho) - i\,n_\ell(\rho)$$

Eq. 3.

$$h^{(1)}_\ell(\rho) \xrightarrow{\rho\to\infty} (-i)\,\frac{e^{i(\rho - \frac{\ell\pi}{2})}}{\rho} = (-i)^{\ell+1}\frac{e^{i\rho}}{\rho}$$

Eq. 4

$$h^{(2)}_\ell(\rho) \xrightarrow{\rho\to\infty} (+i)\,\frac{e^{-i(\rho - \frac{\ell\pi}{2})}}{\rho} = +(+i)^{\ell+1}\frac{e^{-i\rho}}{\rho}$$

* Note that at large distance, scaling behavior of j_l, n_l, $h_l^{(1)}$, $h_l^{(2)}$ are very similar to s-waves apart from phase shifts of "$\frac{l\pi}{2}$".

Eq. 5
$$\begin{cases} j_0(\rho) = \frac{\sin\rho}{\rho}, & n_0(\rho) = -\frac{\cos\rho}{\rho} \\ h_0^{(1)}(\rho) = (-i)\frac{e^{i\rho}}{\rho} & h_0^{(2)}(\rho) = i\frac{e^{-i\rho}}{\rho} \end{cases}$$

<u>why</u> ?

* also note that for $l=0$

$$-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}R_l\right) = k^2 R_l, \qquad R_l = \frac{\chi_l}{r}$$

$$\longrightarrow \quad -\frac{\partial^2}{\partial r^2}\chi_l = k^2\chi_l$$ identical to 1D wave equation.

hence $\chi_{l=0} = \sin\rho$, $-\cos\rho$.