Home set II (Jan 23, 2017)

Solutions

(1.1) To study the relation between phase shifts and bound states, we need to compare the solutions to bound state w.f. and scattering states.

Note $\psi_B(\vec{r}) = h_\ell^{(1)}(i\rho_B)\, Y_\ell^m(\theta,\varphi)$, where Henkle function has been extended to pure imaginary arguments.

$\rho_B = k_B r$, $\quad k_B = \sqrt{2|E_B|} = \frac{1}{a_B}$. $\quad h_\ell^{(1)}(i\rho_B) = j_\ell(i\rho_B) + i n_\ell(i\rho_B)$.

Patching conditions for bound states

$$r < R_0 \qquad\qquad\qquad r > R_0$$

(1) $$r\frac{\partial\tilde{\rho}_B}{\partial r}\left.\frac{j_\ell'(\tilde{\rho}_B)}{j_\ell(\tilde{\rho}_B)}\right|_{r=R_0^-} = r\frac{\partial\rho_B}{\partial r}\left.\frac{j_\ell'(i\rho_B)+i n_\ell'(i\rho_B)}{j_\ell(i\rho_B)+i n_\ell(i\rho_B)}\right|_{r=R_0^+}$$

$$\tilde{\rho}_B = \sqrt{\rho_0^2-\rho_B^2} = \rho_0 - \frac{\rho_B^2}{2\rho_0} + \cdots \qquad (\rho_B \ll \rho_0 = k_0 R_0)$$

$$k_0 = \sqrt{2V_0}$$

Patching conditions for phase-shifted scattering states

(2) $$r\frac{\partial\tilde{\rho}}{\partial r}\left.\frac{j_\ell'(\tilde{\rho})}{j_\ell(\tilde{\rho})}\right|_{r=R_0^-} = r\frac{\partial\rho}{\partial r}\frac{j_\ell'(\rho)\cos\delta_\ell - n_\ell'(\rho)\sin\delta_\ell}{j_\ell(\rho)\cos\delta_\ell - n_\ell(\rho)\sin\delta_\ell}$$

$$\tilde{\rho} = \sqrt{\rho_0^2+\rho^2} = \rho_0 - \frac{\tilde{\rho}^2}{2\rho_0} + \cdots \qquad (\rho \ll \rho_0)$$

$\rho = kr$ at $r = R_0$.

define LNE as $F_\ell(\rho) = \frac{\partial \rho}{\partial t}\, j_\ell(\rho)$, $\rho - kr$

$$F_\ell(\tilde{\rho}_B) = F_\ell(\rho_0)\left[1 + f_\ell(-\rho_B^2)\right]; \quad F_\ell(\tilde{\rho}) = F_\ell(\rho_0)\left[1 + f_\ell \rho^2 + \ldots\right]$$

f_ℓ is a const defined by ℓ-wave and N'th Resonance and depending on the details. It is evident that

$$F_\ell(\tilde{\rho}_B) \approx F_\ell(\tilde{\rho}) \quad \text{if} \quad \rho_B, \rho \to 0.$$

Combining (1) and (2) One can express the phase shift as

$$\boxed{\frac{\tan\delta_\ell - \dfrac{j_\ell'(\rho)}{n_\ell'(\rho)}}{\tan\delta_\ell - \dfrac{j_\ell(\rho)}{n_\ell(\rho)}} = \frac{F_\ell(\tilde{\rho})}{F_\ell(\tilde{\rho}_B)} \cdot \frac{\rho_B \dfrac{n_\ell'(i\rho_B)}{n_\ell(i\rho_B)}}{\rho \dfrac{n_\ell'(\rho)}{n_\ell(\rho)}} \left[\frac{1 + \dfrac{j_\ell'(i\rho_B)}{i n_\ell'(i\rho_B)}}{1 + \dfrac{j_\ell(i\rho_B)}{i n_\ell(i\rho_B)}}\right]}$$

(All arguments ρ, ρ_B are taken at $r = R_0$)
$\rho_B, \rho \ll 1$

Using the small ρ asymptotics for $j_\ell(\rho)$, $n_\ell(\rho)$, $j_\ell(\rho')$, $n_\ell(i\rho)$ we can expand all terms in the equation.

$\ell = 0$ problem is very simple. We are interested in the next order correction in ρ^2 (During my lecture, I did leading order expansion in $O(\rho)$.)

$$\left\{\begin{array}{l} \dfrac{j_0'(\rho)}{n_0'(\rho)} = 0\cdot\rho - \dfrac{\rho^3}{3} + \cdots, \quad \dfrac{j_0(\rho)}{n_0(\rho)} = \rho\left(1 + \dfrac{\rho^2}{3} + \cdots\right) \\[2ex] \dfrac{j_0'(i\rho_B)}{i\,n_0'(i\rho_B)} = 0\cdot\rho_B + \dfrac{\rho_B^3}{3}, \quad \dfrac{j_0(i\rho_B)}{i\,n_0(i\rho_B)} = -\rho_B\left[1 - \dfrac{\rho_B^2}{3}\right] \end{array}\right.$$

$$\left[\begin{array}{l} \rho\,\dfrac{n_0'(\rho)}{n_0(\rho)} = -(1+\rho^2), \quad \rho_B\,\dfrac{n_0'(i\rho_B)}{n_0(i\rho_B)} = -(1-\rho_B^2) \\[2ex] \dfrac{F_0(\tilde\rho)}{F_0(\tilde\rho_B)} = \left[1 + f_0(\tilde\rho^2 - \tilde\rho_B^2) + \cdots\right] \end{array}\right.$$

$f_0 = f_0(k_0 r_0)$ computable for square-well potential

$$\frac{\tan\delta_0 + \frac{\rho^3}{3}}{\tan\delta_0 + \rho\left(1 + \frac{\rho^2}{3}\right)} = \left[1 + f_0(\rho^2 - \rho_B^2)\right]\left[1 - \rho_B^2 + \rho^2\right] \times \left[1 + \frac{\rho_B^3}{3} + \rho_B\left(1 - \frac{\rho_B^3}{3}\right) + \cdots\right]$$

$$\rightarrow \tan\delta_0\left[\rho_B + (f_0 - 1)\rho^2 + (-1)(f_0+1)\rho_B^2 + \cdots\right]$$

$$= -\rho\left(1 + \rho_B + (f_0 - 1)\rho^2 - (f_0+1)\rho_B^2 + \cdots\right]$$

leading term Subleading term

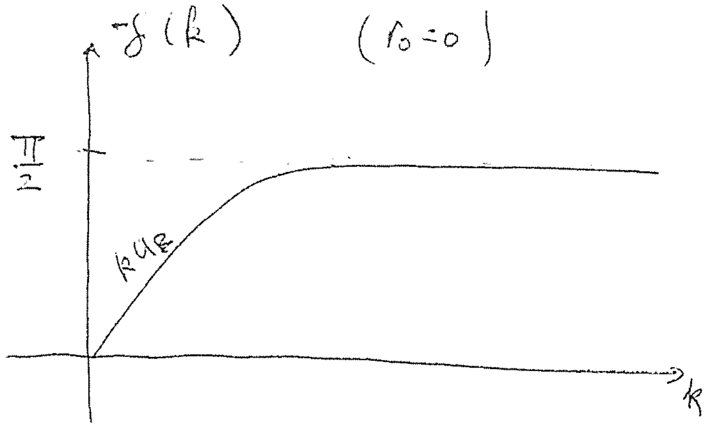
$$\tan\delta_0\left[\rho_B + C_f\rho^2\right] = \rho, \quad \rho_B = k_0 a_B^{-1}, \quad \rho = kR_0$$

$$\boxed{\tan\delta_0 = -\frac{k a_B}{1 + C_f\, k R_0\, k a_B}}$$

$\Rightarrow$ finite Range Effect

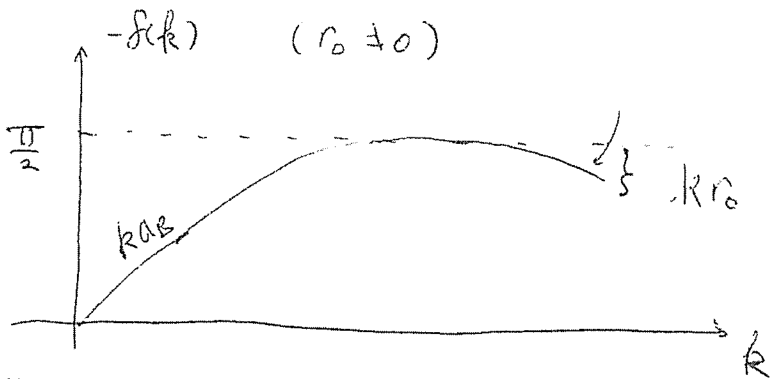
1.2)

$r_0 \to 0$, $\tan\delta_0 = k a_B$ derived before



$ka_B \gg 1$

but $kR_0 \ll 1$, $-\delta(k) = \frac{\pi}{2} - \frac{1}{ka_B}$



following the Reslut,

When $\frac{1}{kR_0} \gg ka_B \gg 1$, $-\delta(k) \simeq \frac{\pi}{2} - \frac{1}{ka_B}$

When $\infty \gg ka_B \gg \frac{1}{kR_0}$, $-\delta(k) \simeq \frac{\pi}{2} - kr_0$

(but $kR_0 \ll 1$)

$\boxed{\text{As far as } kR_0 \ll 1, \quad \delta(k) \text{ at finite } r_0 \simeq \delta(k) \text{ at } r_0 = 0}$

Although $\delta(k) + \frac{\pi}{2}$ at large k behaves very different.

2.1) Asymptotics of $j_\ell(\rho)$, $n_\ell(\rho)$ and $h_\ell^{(1)}(\rho)$ etc.

$$j_\ell(\rho) = \rho^\ell\left(-\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^\ell j_0(\rho), \quad n_\ell(\rho) = \rho^\ell\left(-\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^\ell n_0(\rho)$$

At small ρ, $\quad j_0(\rho) = \frac{\sin\rho}{\rho} \xrightarrow{\rho\to 0} \sum_{n=0}^{\infty} \frac{(-1)^n \rho^{2n}}{(2n+1)!}$

by appling $\left(-\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^\ell$, one obtains

$$j_\ell(\rho) \overset{\rho\to 0}{=} \frac{\rho^\ell}{(2\ell+1)!!}\left[1 - \rho^2\frac{(\ell+1)}{(4\ell+6)\cdots} + \cdots\right]$$

Following the same argument, $\quad n_0(\rho) = -\frac{\cos\rho}{\rho} \xrightarrow{\rho\to 0} \sum_{n=0}^{\infty} \frac{(-1)^n \rho^{2n-1}}{(2n)!}$

$$n_\ell(\rho) \overset{\rho\to 0}{=} -\frac{1}{\rho^\ell}(2\ell-1)!!\left[1 - \frac{\rho^2}{2(2\ell-1)} + \cdots\right]$$
$(\ell = 1, 2, \cdots)$

2.2) $\quad h_\ell^{(1)}(\rho) = \rho^\ell\left(-\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)^\ell h_0^{(1)}(\rho), \quad h_0^{(1)}(\rho) = (-i)\frac{e^{i\rho}}{\rho}$

$$h_\ell^{(1)}(\rho) \overset{\rho\to\infty}{=} \frac{\rho}{\rho}(-i)^{\ell+1}\frac{e^{i\rho}}{\rho} = \frac{1}{\rho}e^{i\rho - i(\ell+1)\frac{\pi}{2}}$$

$$\begin{cases} j_\ell(\rho) = \frac{1}{2}\left(h_\ell^{(1)}(\rho) + h_\ell^{(2)}(\rho)\right) \overset{\rho\to\infty}{=} \frac{1}{\rho}\sin\left(\rho - \frac{\ell\pi}{2}\right) \\ n_\ell(\rho) = \frac{1}{2i}\left(h_\ell^{(1)}(\rho) - h_\ell^{(2)}(\rho)\right) \overset{\rho\to\infty}{=} -\frac{1}{\rho}\cos\left(\rho - \frac{\ell\pi}{2}\right) \end{cases}$$

2.2) near $k \to 0$, $l=1$ partial wave

$$r\frac{\partial\tilde{\rho}}{\partial r}\left.\frac{j_1'(\tilde{\rho})}{j_1(\tilde{\rho})}\right|_{r=R_0^{-1}} = r\frac{\partial\rho}{\partial r}\left.\frac{j_1'(\rho)\cos\delta_{l=1} - n_{l=1}'(\rho)\sin\delta_{l=1}}{j_1(\rho)\cos\delta_1 - n_1(\rho)\sin\delta_1}\right|_{r=R_0}$$

$\tilde{\rho} \doteq k_0 R_0 \ll 1$,

$$j_1(\rho) \xrightarrow{\rho\to 0} \frac{\rho}{3} - \frac{\rho^3}{30}, \quad n_1(\rho) \xrightarrow{\rho\to 0} -\left(\frac{1}{\rho^2} + \frac{1}{2}\right)$$

$$\tilde{\rho}\frac{j_1'(\tilde{\rho})}{j_1(\tilde{\rho})} = 1 - \frac{\tilde{\rho}^2}{15}, \qquad \rho\frac{n_{l=1}'(\rho)}{n_1(\rho)} = -2\left\{1 - \frac{\rho^2}{2} + \cdots\right\}$$

$$\frac{\tan\delta_1 - \frac{j_1'(\rho)}{n_1'(\rho)}}{\tan\delta_1 - \frac{j_1(\rho)}{n_1(\rho)}} \cdot \underbrace{\rho\frac{n_{l=1}'(\rho)}{n_1(\rho)}} = \tilde{\rho}\frac{j_1'(\tilde{\rho})}{j_1(\tilde{\rho})} = 1 - \frac{\tilde{\rho}^2}{15}$$

$$\frac{j_1'(\rho)}{n_1'(\rho)} = +\frac{\rho^3}{6}, \qquad \frac{j_1(\rho)}{n_1(\rho)} = -\frac{\rho^3}{3}$$

$$\frac{\tan\delta_1 - \frac{\rho^3}{6}}{\tan\delta_1 + \frac{\rho^3}{3}} = -\frac{1}{2}\left[1 + \frac{\rho^2}{2} - \frac{\tilde{\rho}^2}{15} + \cdots\right]$$

$$\to \tan\delta_1 = +\frac{\rho^3\tilde{\rho}^2}{135} = \frac{1}{135}(kR_0)^3\underbrace{k_0^2R_0^2}_{B} = k^3 V_s$$

$$V_s = \frac{R_0^3(k_0^2R_0^2)}{135} = \frac{R_0^3 B}{135}$$

2.3) if $B \ll 1$, k-dependence mainly coming from

$$\tilde{\rho}^2 = B + \rho^2 = B\left[1 + \frac{k^2 R_0^2}{B} + O(k^2 R_0^2)\right] \cdots$$

indeed, $\tan\delta_1(B, \rho_0) = V_3 k^3 \left[1 + \alpha \rho_0^2 + \cdots\right]$

$$\alpha = -\frac{13}{30}\frac{1}{B} \qquad \text{Note } |\alpha| \propto \frac{1}{B} \gg 1.$$

the rest dependence of order of ρ_0^2.

2.4) For a bound state to appear,

$$\tilde{\rho}_0 \frac{j_\ell'(\tilde{\rho}_0)}{j_\ell(\tilde{\rho}_0)} = \rho_0 \frac{\bar{h}_\ell'(\rho_0)}{\bar{h}_\ell(\rho_0)}, \qquad \bar{h}_\ell(\rho) \sim h_\ell^{(1)}(i\rho)$$

Real bound state wave functions

$$\bar{h}_\ell(\rho) = \rho^\ell \left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}\right]^\ell \bar{h}_0(\rho), \qquad h_0(\rho) = \frac{e^{-\rho}}{\rho}$$

For instance, $\bar{h}_1(\rho) = \dfrac{e^{-\rho}}{\rho^2} + \dfrac{e^{-\rho}}{\rho}$

$$\tilde{\rho}_0 = \sqrt{2(V_0 - E_B) R_0^2} = \sqrt{B - \rho_0^2}, \qquad \rho_0 = k_B r = r/a_B$$

$$E_B = \frac{k_B^2}{2} \leftarrow \text{binding energy.}$$

by taking $k_B \to 0$ or $a_B \to \infty$,

$$\rho_0 \frac{\bar{h}_\ell'(\rho_0)}{\bar{h}_\ell(\rho_0)} \doteq -(\ell+1) \quad \text{so that}$$

$$\tilde{\tilde{\rho}}_0 \frac{j_\ell'(\tilde{\tilde{\rho}}_0)}{j_\ell(\tilde{\tilde{\rho}}_0)} + (\ell+1) = 0, \qquad \tilde{\tilde{\rho}}_0 = \sqrt{B}$$

For $l=1$, this means

$$\left[\tilde{\rho}_0^{\,2}\, j_1(\tilde{\rho}_0)\right]' = 0 \quad \Leftarrow \text{bound state equation for } a_B \to \infty .$$

Giving that $j_1(\tilde{\rho}_0) = -\frac{\partial}{\partial\rho}\frac{\sin\rho}{\rho} = \frac{\sin\tilde{\rho}_0}{\tilde{\rho}_0^2} - \frac{\cos\tilde{\rho}_0}{\tilde{\rho}_0}$

$\tilde{\rho}_0^{\,2} j_1(\tilde{\rho}_0) = \sin\tilde{\rho}_0 - \tilde{\rho}_0\cos\tilde{\rho}_0$. So the condition is again equivalent to $\boxed{\sin\tilde{\rho}_0 = 0}$

or $B^* = \sqrt{2V_0R_0} = \pi,\ 2\pi,\ 3\pi \ldots$

Note that $h_0(\rho) = \frac{1}{\rho} + \sum_{n=1}^{\infty} \frac{(-1)^n \rho^{n-1}}{n!}$

$$\overset{\rho\to 0}{=} \frac{1}{\rho} - 1 + O(\rho)$$

$$= \frac{1}{\rho}\left[1 - \rho^1 + \cdots\right]$$

However, $\bar{h}_l(\rho) = \rho^l\left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}\right]^l h_0(\rho)$

$l \geq 1$

$$\overset{\rho\to 0}{=} \frac{(2l-1)!!}{\rho^{l+1}} + \frac{(-1)^l}{(2l)!!}\rho^{l-1}$$

$$= \frac{(2l-1)!!}{\rho^{l+1}}\left[1 + \underbrace{\frac{(-1)^l \rho^{l\cdot 2}}{(2l)!!\,(2l-1)!!}}_{\text{Lowest correction } \rho^{2l}} + \cdots\right]$$

2.5)

$l=1$ case, $\quad \bar{h}_{l=1}(\rho) = \frac{1}{\rho^2}\left[1 - \frac{\rho^2}{2} + \cdots\right]$

$$\bar{h}'_l(\rho) = -\frac{2}{\rho^3}\left[1 + 0. + O(\rho^2) + \cdots\right]$$

The bound state equation

$$\tilde{\rho}_0 \frac{j_l'(\tilde{\rho}_0)}{j_l(\tilde{\rho}_0)} = \rho_0 \frac{\bar{h}_l'(\rho_0)}{\bar{h}_l(\rho_0)}$$

Expanded around B^* where $\rho_0 = 0$ or $a_B \to \infty$,

$$\left[2 + \tilde{\rho}_0 \frac{j_l'(\tilde{\rho}_0)}{j_l(\tilde{\rho}_0)}\right] = \rho_0^2 A$$

$$\to 2 + \tilde{\tilde{\rho}}_0 \frac{j_l'(\tilde{\tilde{\rho}}_0)}{j_l(\tilde{\tilde{\rho}}_0)} + B_A(\rho_0^2) = A\rho_0^2 \to \underbrace{2 + \tilde{\tilde{\rho}}_0 \frac{j_l'(\tilde{\tilde{\rho}}_0)}{j_l(\tilde{\tilde{\rho}}_0)}} = C\rho_0^2$$

$$\tilde{\tilde{\rho}}_0^2 = B = \underbrace{B - B^*} + B^*,$$

Expanding LHE around B^* yields: $\quad 0 + B - B^* = C k_B^2 R_0^2$

or $\quad k_B^2 \approx (B - B^*)$, $\quad a_B \sim \frac{1}{(B-B^*)^{1/2}}$

The power here is "$1/2$" unlike in the S-wave case,

$$a_B \sim \frac{1}{[B - B^*]^{1}}.$$

Prob. 3.

3.1) Following our lecture and prob. 2, it is possible to show

$$\tan\delta_\ell(B_0, \rho_0) = \frac{1 - \tilde{\rho}_0 \dfrac{j_\ell'(\tilde{\rho}_0)}{j_\ell(\tilde{\rho}_0)} \dfrac{1}{\rho} \dfrac{j_\ell(\rho_0)}{j_\ell'(\rho_0)}}{\dfrac{n_\ell'}{j_\ell'(\rho_0)} - \tilde{\rho} \dfrac{j_\ell'(\tilde{\rho}_0)}{j_\ell(\tilde{\rho}_0)} \dfrac{1}{\rho} \dfrac{j_\ell(\rho_0)}{j_\ell'(\rho_0)} \dfrac{n_\ell(\rho_0)}{j_\ell(\rho_0)}}$$

$$\text{or} = \underbrace{\frac{j_\ell'(\rho_0)}{n_\ell'(\rho_0)}}_{-\frac{1}{(2\ell+1)!!(2\ell-1)!!}\rho_0^{2\ell+1}} \left[\frac{1 - \tilde{\rho}_0 \dfrac{j_\ell'(\tilde{\rho}_0)}{j_\ell(\tilde{\rho}_0)} \dfrac{1}{\rho} \dfrac{j_\ell(\rho_0)}{j_\ell'(\rho_0)}}{1 - \tilde{\rho}_0 \dfrac{j_\ell'(\tilde{\rho}_0)}{j_\ell(\tilde{\rho}_0)} \dfrac{1}{\rho_0} \dfrac{n_\ell(\rho_0)}{n_\ell'(\rho_0)}} \right]$$

$$= (kR_0)^{2\ell+1} \mathcal{V}_\ell(B, \rho_0)$$

$$\mathcal{V}_\ell(B, \rho_0) = -\frac{1}{(2\ell+1)!!}\frac{1}{(2\ell-1)!!} \left[\frac{1 - \tilde{\rho}_0 \dfrac{j_\ell'(\tilde{\rho}_0)}{j_\ell(\tilde{\rho}_0)} \dfrac{1}{\rho} \dfrac{j_\ell(\rho_0)}{j_\ell'(\rho_0)}}{1 - \tilde{\rho}_0 \dfrac{j_\ell'(\tilde{\rho}_0)}{j_\ell(\tilde{\rho}_0)} \dfrac{1}{\rho_0} \dfrac{n_\ell(\rho_0)}{n_\ell'(\rho_0)}} \right]$$

denominator

$$= 1 - \tilde{\rho}_0 \frac{j_\ell'(\tilde{\rho}_0)}{j_\ell(\tilde{\rho}_0)} \frac{1}{\rho_0} \frac{n_\ell(\rho_0)}{n_\ell'(\rho_0)} \overset{\rho_0 \to 0}{=} (\ell+1) + \tilde{\rho}_0 \frac{j_\ell'(\tilde{\rho}_0)}{j_\ell(\tilde{\rho}_0)} = 0$$

$$\text{or} \quad \left[\tilde{\rho}_0^{\,\ell+1} j_\ell(\tilde{\rho}_0) \right]' = 0 \quad \text{Resonance Condition}$$

3.2)

Note that $\dfrac{\tilde{\rho}_0\, j_\ell'(\tilde{\rho}_0)}{j_\ell(\tilde{\rho}_0)} = \rho_0 \dfrac{\overline{h}_\ell'(\rho_0)}{h_\ell(\rho_0)} \overset{\rho_0 \to 0}{=} -(\ell+1)$

So: $\rho_0 \to 0$, $\tilde{\rho}_0 \to \hat{\tilde{\rho}}_0$ and above Equation evolves into : $\left[\hat{\tilde{\rho}}_0^{\,\ell+1}\, j_\ell(\hat{\tilde{\rho}}_0)\right]' = 0$ for $B = B^*$.

identical to 3.1). Resonance condition

*Recall $h_\ell(\rho_0) \overset{\rho_0 \to 0}{=} \dfrac{(2\ell-1)!!}{\rho_0^{\ell+1}}$ as $\underline{n_\ell}(\rho_0)$

3.3) Following the same tactic of 2.5)

when $a_B \to \infty$, the bound state equation can be expanded around B^*, and $\rho_0 \to 0$ at the same time.

$$LHSE = \tilde{\rho}_0 \frac{j_\ell'(\tilde{\rho}_0)}{j_\ell(\tilde{\rho}_0)} = \left.\tilde{\rho}_0 \frac{j_\ell'(\tilde{\rho}_0)}{j_\ell(\tilde{\rho}_0)}\right|_{B=B^*} + C_1(B-B^*)$$

$$+ C_2(\underline{\rho_0}^2) \qquad \left(\text{Note } \tilde{\rho}_0 = [B+\rho_0^2]^{1/2} = B + \frac{\rho_0^2}{2B} + \cdots\right.$$

$$RHE = -(\ell+1) + C_3 \rho_0^{2\ell}, \qquad \ell > 1$$

$$LHE = RHE \longrightarrow \rho_0^2 \sim (B-B^*) \text{ or } \frac{1}{a_B} \sim (B-B^*)^{\frac{1}{2}}.$$

Note: $\left.\tilde{\rho}_0 \dfrac{j_\ell'(\tilde{\rho}_0)}{j_\ell(\tilde{\rho}_0)}\right|_{B=B^*} + (\ell+1) = 0$

Conclusion for all $l \neq 0$ or $l = 1, 2, 3 \ldots$

$$a_B \sim \frac{1}{(B - B^*)^{1/2}}$$

a_B — bound state size

$k_B = \frac{1}{a_B}$, $E_B = \frac{k_B^2}{2}$

For $l = 0$, S-wave

$$a_B \sim \frac{1}{[B - B^*]^1}$$