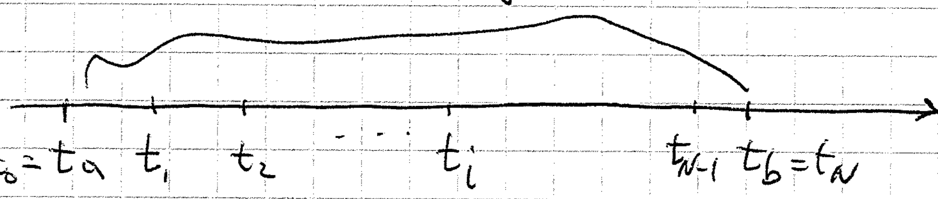
1

Solutions

$$\epsilon = \frac{t_b - t_a}{N}$$

Prob. 0.



$$\langle \vec{x}_{i+1} | e^{-iH_0\epsilon} | \vec{x}_i \rangle = \sum_{\vec{p}} \langle \vec{x}_{i+1} | e^{-iH_0\epsilon} | \vec{p} \rangle \langle \vec{p} | \vec{x}_i \rangle \qquad H_0 = \frac{p^2}{2}$$

t_{i+1} $\quad$ t_i

Summation over $\vec{p}$ can be carried out using analytic continuations

$$U(\vec{x}_{i+1}, t_{i+1}; \vec{x}_i, t_i) = A^3 e^{i\frac{(\vec{x}_{i+1} - \vec{x}_i)^2}{2\epsilon}}$$

General case in 3D, one can show that

$$\int \mathcal{D}\vec{x}(t) = A^3 \underbrace{\int A^3 d^3\vec{x}_1 \cdots A^3 d^3\vec{x}_{N-1}}_{N-1 \text{ integrals}} \qquad A(\epsilon) = \left[\frac{M}{2\pi i \epsilon}\right]^{1/2}$$

derived during lecture

$$S(\{\vec{x}(t)\}) = \int_{t_a}^{t_b} \left[\frac{1}{2}\dot{\vec{x}}(t)^2 - V(\vec{x}(t))\right] dt$$

$$U(\vec{x}_b, t_b; \vec{x}_a, t_a) = \int_{\vec{x}(t_a)=\vec{x}_a}^{\vec{x}(t_b)=\vec{x}_b} \mathcal{D}\vec{x}(t)\, e^{iS(\{\vec{x}(t)\})}$$

Prob. 1 For this interaction,

$$\mathcal{U}(x_b, t_b; x_a, t_a) = A(t_b - t_a)\, e^{iS(\{x_{cl}(t)\})}$$

$$x(t) = x_a + v_0(t - t_a) + \frac{1}{2}\frac{f}{m}(t - t_a)^2$$

$$v_0 = \frac{x_b - x_a - \frac{1}{2}\left(\frac{f}{m}\right)(t_b - t_a)^2}{t_b - t_a}$$

[Set $t_a = 0$ to simplify Notations]

$$\mathcal{L}(\{x_{cl}\}) = \frac{m}{2}\left(v_0 + \frac{f}{m}t\right)^2 + f\left(x_a + v_0 t + \frac{1}{2}\left(\frac{f}{m}\right)t^2\right)$$

$$\int_{t_a=0}^{t_b} \mathcal{L}(\{x_{cl}(t)\})\, dt = \frac{m}{2}\left[\frac{x_b - x_a}{t_b - t_a}\right]^2 + \frac{1}{2} f (x_b + x_a)(t_b - t_a) - \frac{m}{2}\left(\frac{f}{m}\right)^2 (t_b - t_a)^3$$

identical to Eq. 5.4.31. $A(t_b - t_a)$ can also be computed easily using the analytic continuation and the Result is

$$A(t_b - t_a) = \left[\frac{M}{2\pi i (t_b - t_a)}\right]^{1/2}$$

Prob. 2.
Exercise 8.6.2

Set $t_a = 0$

$$x_{cl}(t) = x_1 \cos\omega t + x_2 \sin\omega t \qquad x_{cl}(t_a = 0) = x_a$$

$$x_{cl}(t_b) = x_b$$

$$= x_a \cos\omega t + \frac{x_b - x_a \cos\omega t_b}{\sin\omega t_b} \sin\omega t$$

$$S(\{x_{cl}(t)\}) = \frac{M\omega^2}{2}\left[(x_1^2 \mp x_2^2)(-)\int_0^{t_b} \cos 2\omega t\, dt - 2x_1 x_2 \int_0^{t_b} \sin 2\omega t\, dt\right]$$

$$= \frac{M\omega^2}{2}\left[(x_1^2 \mp x_2^2)(+\sin 2\omega t_b) + 2 x_1 x_2 (\cos 2\omega t_b - 1)\right] \cdot \frac{1}{2\omega}$$

$$\begin{cases} -x_1^2 + x_2^2 = \dfrac{\cos 2\omega t_b\, x_a^2 + x_b^2 - 2 x_b x_a \cos\omega t_b}{\sin^2 \omega t_b} \\[2ex] 2 x_1 x_2 = \dfrac{2 x_a x_b - 2 x_a^2 \cos\omega t_b}{\sin^2 \omega t_b} \end{cases}$$

$$S(\{x_{cl}(t)\}) = \frac{M\omega}{2\sin\omega(\underbrace{t_b - t_a}_{})}\left[(x_a^2 + x_b^2)\cos\omega(t_b - \underbrace{t_a}_{=0}) - 2x_a x_b\right]$$

$t_a = 0$ (If t_a set to be zero)

Prob. 3.

1) $$A(t) = \left[\frac{M\omega}{2\pi i \sin\omega t}\right]^{1/2}$$

Computed using complex analyses

$$= \left[\frac{M\omega}{\pi\hbar}\right]^{1/2} e^{-i\frac{\omega t}{2}} \underbrace{\left[1 + \frac{1}{2} e^{-i2\omega t} + \frac{3}{8} e^{-i4\omega t} + \cdots\right]}_{\sum_{n=0}^{\infty} \frac{1}{n!} \frac{(2n-1)!!}{2^n} e^{-i2n\omega t}}$$

$$\underset{\substack{\| \\ x_b}}{\overset{}{U}}(\overset{\underset{\|}{t_b}}{}x', t'; x, t) \quad (x' = x_b,\ t' = t_b,\ x = x_a,\ t = t_a)$$

$$U(x', t'; x, t) \underset{x'=x=0}{=} \sum_n \psi_n^*(0)\psi_n(0)\, e^{-i\frac{\omega t}{2}} e^{-in\omega t}$$

$\psi_n(0) = 0$ if $n =$ odd

$$\underset{x'=x=0}{=} e^{-i\frac{\omega t}{2}} \sum_n \psi_{2n}^*(0)\psi_{2n}(0)\, \underbrace{e^{-2in\omega t}}$$

Consisten with $A(t)$

2) $$S(\{x_a(t)\}) = \frac{iM\omega}{2\sin\omega t_b}\left[2x_b^2\cos\omega_b t - 2x_b^2\right], \quad x_b = x_a$$

$$= -M\omega x_b^2 \left[\frac{1 - e^{-i\omega t_b}}{1 + e^{-i\omega t_b}}\right]$$

For the discussion on $n = 0, 1$, it is sufficient to keep up to linear term in $e^{-i\omega t_b}$.

$$u(x, t; x, 0) \doteq \frac{M\omega}{\pi\hbar} e^{-i\frac{\omega t}{2}} \left[e^{-M\omega x^2} + 2M\omega x^2 e^{-M\omega x^2} e^{-i\omega t} + \cdots\right]$$

Comparing with exact expression for $U(x,t;x,0)$ we identify

$$E_0 = \frac{\omega}{2}, \qquad |\psi_0(x)|^2 = \left[\frac{M\omega}{\pi}\right]^{1/2} e^{-M\omega X^2}$$

$$E_1 = \frac{3\omega}{2}, \qquad |\psi_1(x)|^2 = \left[\frac{M\omega}{\pi}\right]^{1/2} (2M\omega X^2)\, e^{-M\omega X^2}$$