Homework Set 8

Solutions

1)

$$\pi_\alpha = P_\alpha - \frac{q}{c} A_\alpha$$

$$\left(\vec{\pi} \times \vec{\pi}\right)_\alpha = \epsilon_{\alpha\beta\gamma} \pi_\beta \pi_\gamma = \epsilon_{\alpha\beta\gamma} P_\beta \left(-\frac{q}{c} A_\gamma\right)$$

$$= -i\frac{q}{c} \epsilon_{\alpha\beta\gamma} \nabla_\beta A_\gamma = \frac{q B_\alpha}{c}$$

2)

$$|\psi\rangle = \begin{bmatrix} \chi \\ \frac{\vec{\sigma}\cdot(\vec{P}-\frac{q}{c}\vec{A})}{E+mc^2}\chi \end{bmatrix} \qquad \begin{bmatrix} -\frac{\vec{\sigma}\cdot(\vec{P}-\frac{q}{c}\vec{A})}{E+mc^2}\chi \\ \chi \end{bmatrix}$$

$E > 0$

For χ,

$$(E - mc^2)|\chi\rangle = \frac{\left[\vec{\sigma}\cdot(\vec{P}-\frac{q}{c}\vec{A})\right]^2}{E+m} |\chi\rangle$$

2-Component spinor

$$= \frac{(P - \frac{q}{c}\vec{A})^2 + i\epsilon_{\alpha\beta\gamma}(P_\beta - \frac{q}{c}A_\beta)(P_\gamma - \frac{q}{c}A_\gamma)\sigma_\gamma}{E+mc^2}$$

The second term leads to $-\epsilon_{\alpha\beta\gamma}[\nabla_\beta, A_\gamma]\frac{q}{c}\sigma_\gamma$

$$= -\frac{q}{c}\vec{\sigma}\cdot\vec{B}$$

Since $Q = P_x - \frac{q}{c}A_x = \frac{\nabla_x}{i} + \frac{q}{2c}By \quad P = P_y - \frac{q}{c}A_y = \frac{\nabla_y}{i} - \frac{q}{2c}Bx$

$[Q, P] = i\frac{qB}{c}$ are conjugate variables like $[x, p]$

The spectrum for $\frac{Q^2}{2} + \frac{P^2}{2}$ shall be the same as

For Harmonic oscillators,

$$H_{QP} = \frac{Q^2}{2} + \frac{P^2}{2} = \left(n + \tfrac{1}{2}\right)\frac{qB}{c}$$

[To see this, further introducing $Q = \tilde{Q}\left(\frac{qB}{c}\right)^{1/2}$, $P = \tilde{P}\left(\frac{qB}{c}\right)^{1/2}$.

$$H_{QP} = \frac{qB}{c}\left[\frac{\tilde{Q}^2}{2} + \frac{\tilde{P}^2}{2}\right], \quad \underbrace{[\tilde{Q}, \tilde{P}] = i}_{\text{Harmonic oscillator with } m=\omega=1}$$

SO:

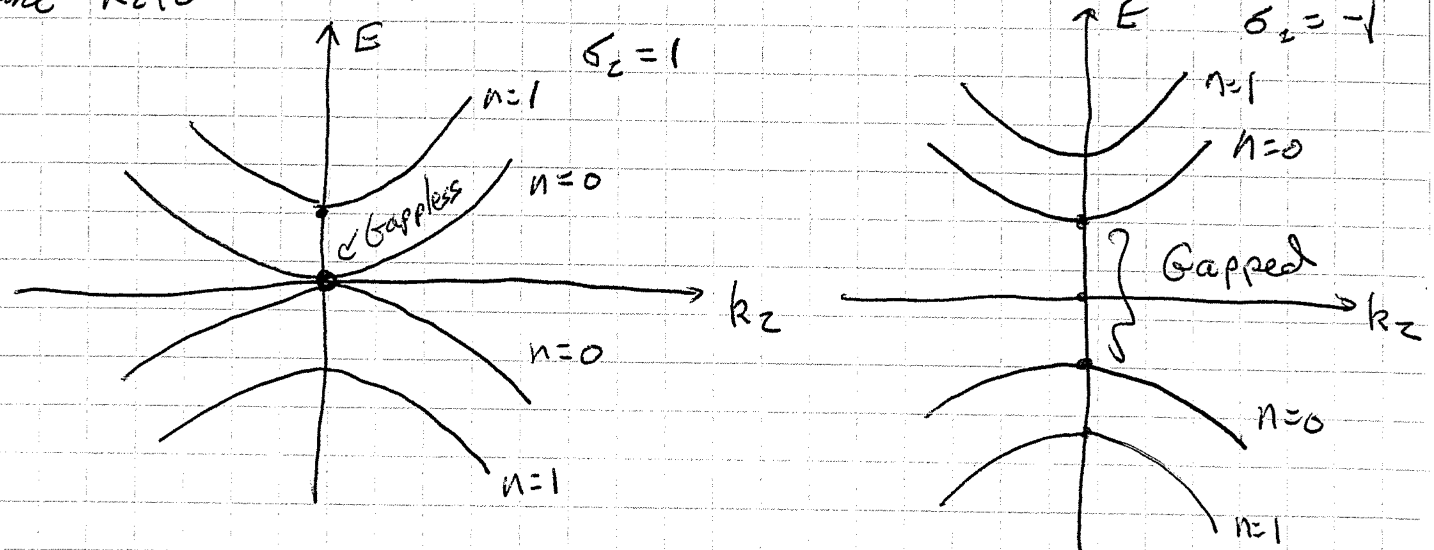
$$\left(E^2 - (\overset{c=1}{m c^2})^2\right)\chi = \left[k_z^2 + (2n+1-\sigma_z)\frac{qB}{c}\right]\chi$$

The spectrum, $E = \pm\sqrt{m^2 + \left[k_z^2 + (2n+1-\sigma_z)\frac{qB}{c}\right]}$

$$\boxed{E = E(k_z, n, \sigma_z)}$$

$\underbrace{n = 0, 1, 2, 3 \ldots}_{\text{Harmonic quantum Number}}$ $\quad\underbrace{\sigma_z = 1, -1}_{\text{Spin along z-direction.}}$

Take $k_z \neq 0$ but $m=0$, massless limit (Schematics)



In the SI unit,

$$E = \pm\sqrt{m^2 c^4 + k_z^2 c^2 + (2n+1-\sigma_z)\frac{qB}{m}\cdot mc^2}, \quad \frac{qB}{m} = \hbar\omega_B$$