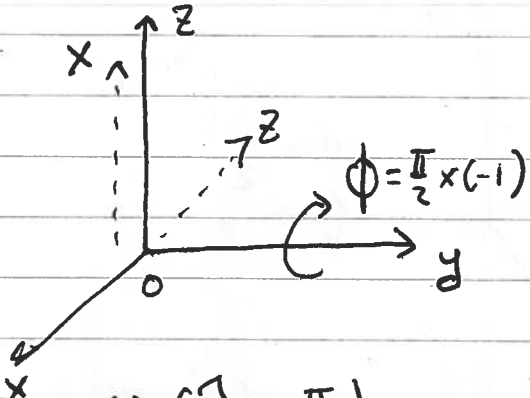
Prob I: α-β Representation

$$i\partial_t \psi = \begin{bmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p}, & -m \end{bmatrix}\psi, \qquad H_0 = m\,\tau_z \otimes \mathbb{1} + \tau_x \otimes \vec{\sigma}\cdot\vec{p}$$

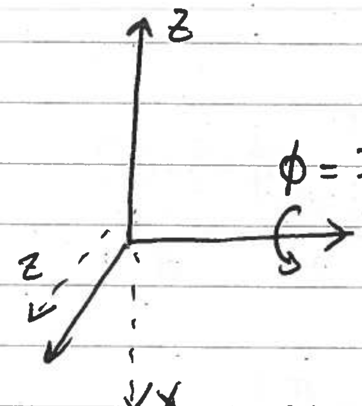
Consider a Rotation in "τ-space" defined as

$$U_\tau(\hat{n}, \Phi) = e^{-i\frac{\vec{\tau}}{2}\otimes\mathbb{1}\cdot\hat{n}\,\Phi}$$



$U_\tau(\hat{e}_y, -\frac{\pi}{2})$

Not this one.



$U_\tau(\hat{e}_y, \frac{\pi}{2})$

This is the one you shall idenitify!

$$\begin{cases} U_\tau(\frac{\pi}{2})\,\tau_x\otimes\vec{\sigma}\cdot\vec{p}\;U_\tau^\dagger(\frac{\pi}{2}) = -\tau_z\otimes\vec{\sigma}\cdot\vec{p} \\ U_\tau(\frac{\pi}{2})\,\tau_z\otimes\mathbb{1}\;U_\tau^\dagger(\frac{\pi}{2}) = \tau_x\otimes\mathbb{1} \end{cases}$$

$$\tilde{H} = U_\tau(\tfrac{\pi}{2})\,H\,U_\tau^\dagger(\tfrac{\pi}{2}) = -\tau_z\otimes\vec{\sigma}\cdot\vec{p} + \tau_x\otimes\mathbb{1}\,m$$

multiplying

$$\rightarrow (\tau_x\otimes\mathbb{1}) \times \left[\, i\partial_t + \tau_z\otimes\vec{\sigma}\cdot\vec{p} - \tau_x\otimes\mathbb{1}\,m \right]\tilde{\psi} = 0$$

$$\rightarrow \left[i\gamma^\mu\partial_\mu - m \right]\tilde{\psi} = 0,$$ Consistent with γ-Rep.

$$\gamma^0 = \begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1}, & 0 \end{bmatrix} = \tau_x\otimes\mathbb{1}, \quad \vec{\gamma} = \begin{bmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma}, & 0 \end{bmatrix} = +i\,\tau_y\otimes\vec{\sigma}.$$

Prob 2. $P_{R,L} = \frac{1 \pm \tau_x \otimes \mathbb{1}}{2}$ ($\sim \gamma_5$), $P_R^2 = P_R$, $P_L^2 = P_L$

$\underbrace{P_R + P_L = \mathbb{1}}_{\text{completeness}}$ $\quad P_R P_L = 0$

$$\psi = (P_R + P_L)\psi = \psi_R + \psi_L$$

Dirac Spinor

applying P_R, P_L to the Dirac Equation taking into account

$$P_R \tau_x \otimes \mathbb{1} = P_R, \quad P_L \tau_x \otimes \mathbb{1} = -P_L, \quad P_R \tau_z = \tau_z P_L$$

$$\begin{cases} i\partial_t \psi_L = -\mathbb{1} \otimes \vec{\sigma}\cdot\vec{p}\, \psi_L + \tau_z \otimes \mathbb{1}\, m\, \psi_R \\ i\partial_t \psi_R = \mathbb{1} \otimes \vec{\sigma}\cdot\vec{p}\, \psi_R + \tau_z \otimes \mathbb{1}\, m\, \psi_L \end{cases}$$

Massless limit leads to Weyl fermions.

$$\boxed{i\partial_t \psi_{R,L} = \pm \mathbb{1} \otimes \vec{\sigma}\cdot\vec{p}\, \psi_{R,L}}$$

$$\psi_R = \frac{1}{\sqrt{2}}\begin{bmatrix} \chi_R \\ \chi_R \end{bmatrix}$$

$$\psi_L = \frac{1}{\sqrt{2}}\begin{bmatrix} \chi_L \\ -\chi_L \end{bmatrix}$$

Prob. 3.

$$\begin{cases} \rho_L = \psi_L^\dagger \psi_L, & j_L = -\psi_L^\dagger \vec{\sigma} \otimes \mathbb{1} \psi_L \\ \rho_R = \psi_R^\dagger \psi_R, & j_R = \psi_R^\dagger \vec{\sigma} \otimes \mathbb{1} \psi_R \end{cases}$$

$$\partial_t(\rho_R - \rho_L) + \vec{\nabla}\cdot\left(\psi_R^\dagger \mathbb{1} \otimes \vec{\sigma}\, \psi_R + \psi_L^\dagger \vec{\sigma} \otimes \mathbb{1}\, \psi_L\right)$$

$$= \left(\psi_R^\dagger \frac{\tau_z}{i} \psi_L \overset{(-)}{+} \psi_L^\dagger \frac{\tau_z}{i} \psi_R\right) m$$

$$\rho_5 = \rho_R - \rho_L = \psi^\dagger \gamma_5 \psi, \quad \vec{J}_5 = \vec{j}_R - \vec{j}_L = \psi^\dagger \gamma_5 \vec{\alpha} \psi$$

$$\gamma_5 = \tau_x \otimes \mathbb{1}$$

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$$\text{Source term} = \psi^\dagger\left(P_R \frac{\tau_z}{i} P_L - P_L \frac{\tau_z}{i} P_R\right)\psi = -2\,\psi^\dagger \underbrace{\tau_y \otimes \mathbb{1}}_{\Gamma}\,\psi$$

$$\tau_y \otimes \mathbb{1} = \begin{bmatrix} 0 & -i\mathbb{1} \\ i\mathbb{1} & 0 \end{bmatrix}$$

Prob. 4. As the mass term $\tau_z \otimes \mathbb{1}$ commute with $\mathbb{1} \otimes \bar{\sigma}\cdot\bar{p}$, it doesn't "scramble spinors"

2-component χ.

Let us set $\vec{\sigma}\cdot\hat{p}\,\chi_+ = +\chi_+$ with spin pointing along $\vec{p}$.

Two eigen states with $\vec{\sigma}\cdot\hat{p} = +1$ are following Dirac Equation: $N = 1/\sqrt{1+A^2}$.

$$\underbrace{\psi_{+,+}}_{\text{frequency},\ \text{spin}}(\bar{p}) = N\begin{bmatrix} \chi_+ \\ \frac{p}{|E_p| + m}\chi_+ \end{bmatrix}, \quad \underbrace{\psi_{-,+}}_{\text{freq.},\ \text{spin}}(\vec{p}) = N\begin{bmatrix} -\frac{p}{E_p + m}\chi_+ \\ \chi_+ \end{bmatrix}$$

$$\begin{cases} \psi(t=0) = \frac{1}{\sqrt{2}}\begin{bmatrix} \chi_+ \\ \chi_+ \end{bmatrix} = \frac{1}{\sqrt{2}}\frac{1+A}{\sqrt{1+A^2}}\psi_{+,+}(\vec{p}) + \frac{1}{\sqrt{2}}\frac{1-A}{\sqrt{1+A^2}}\psi_{-,+}(\vec{p}) \\ \psi(t\neq 0) = \frac{1}{\sqrt{2}}\frac{(1+A)}{\sqrt{1+A^2}}\psi_{+,+}(\vec{p})e^{-iE_+t} + \frac{1}{\sqrt{2}}\frac{(1-A)}{\sqrt{1+A^2}}\psi_{-,+}(\vec{p})e^{-iE_-t} \end{cases}$$

$$Q_5 = \psi^\dagger \underbrace{\overset{P_R - P_L}{\tau_z \otimes \mathbb{1}}}_{\gamma_5}\psi = \psi_H^\dagger\psi_H - \psi_L^\dagger\psi_L, \quad A = \frac{p}{|E_p| + m} = \frac{p}{|E_p| + m}$$

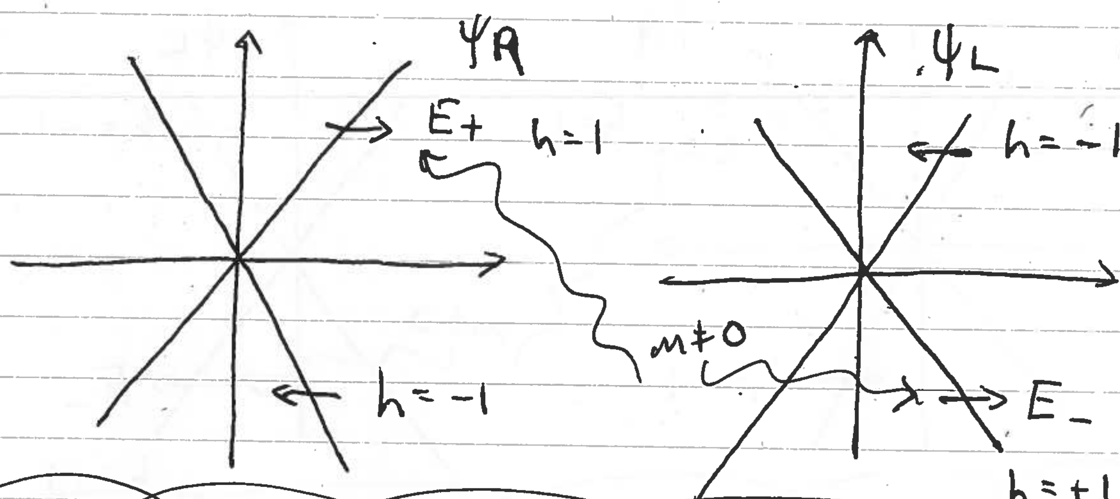
$$= \frac{4A^2}{(1+A^2)^2} + \frac{(1-A^2)^2}{(1+A^2)^2}\cos\omega t.$$

$$[\omega = E_+ - E_- = 2|E_p|]$$

Note $A = 1$ and $1 - A^2 = 0$ when $m \to 0$

Note in this problem, only χ_+ for $\vec{\sigma}\cdot\hat{p} = \mathbb{1}$ states are involved. χ_- states are not involved.

$\chi_+^\dagger \chi_- = 0$. and the mass term doesn't alter the spin or Helicity of the states.



$\psi_{-,-}(\vec{p})$, $\psi_{+,-}(\vec{p})$ are not involved

Two states with $\chi_-(\vec{p})$

You can also choose to work with Weyl Rep.

$$H_D' = \begin{bmatrix} \vec{\sigma}\cdot\vec{p} & 0 \\ 0 & -\vec{\sigma}\cdot\vec{p} \end{bmatrix} + \begin{bmatrix} 0 & -m \\ -m & 0 \end{bmatrix}$$

$$= \tau_z \otimes \vec{\sigma}\cdot\vec{p} + (-\tau_x) \otimes \mathbb{1}\, m.$$

Here $\psi_H = \begin{bmatrix} \chi_\pm \\ 0 \end{bmatrix}$ $\psi_L = \begin{bmatrix} 0 \\ \chi_\pm \end{bmatrix}$, $t=0$, $\psi = \psi_H(\chi_+)$

You will obtain the same result