Solutions to HW Set V

Prob. I.

$$\vec{\alpha} = \begin{bmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{bmatrix} = \tau_x \otimes \vec{\sigma}$$

$\tau_x \to$ P–Antip space, $\vec{\sigma} \to$ Spin Space

$$\beta = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} = \tau_z \otimes \mathbb{1}$$

$$\{\tau_x, \tau_z\} = 0$$

$$\{\alpha_\alpha, \alpha_\beta\} = \{\tau_x \otimes \sigma_\alpha, \tau_x \otimes \sigma_\beta\} = \{\sigma_\alpha, \sigma_\beta\} = 2\delta_{\alpha\beta} \cdot \mathbb{1}$$

$\mathbb{1}$: 4x4 Unity Matrix

$$\{\alpha_\alpha, \beta\} = \{\tau_x, \tau_z\}\sigma_\alpha = 0, \quad \{\beta, \beta\} = 2\mathbb{1}$$

$\mathbb{1}$: 4x4 unity Matrix

Prob. II

$$\psi_{+,\sigma}(\vec{p}) = \mathcal{N}\begin{bmatrix} \chi \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{|E_p|+m}\chi \end{bmatrix}, \quad \psi_{-,\sigma}(\vec{p}) = \mathcal{N}\begin{bmatrix} -\dfrac{\vec{\sigma}\cdot\vec{p}}{|E_p|+m}\phi \\ \phi \end{bmatrix}$$

$\mathcal{N}$: Normalization.

choose $\underbrace{\vec{\sigma}\cdot\dfrac{\vec{p}}{|\vec{p}|}}_{\text{Helicity operator}} \chi_\pm(\hat{p}) = \pm\chi_\pm(\hat{p})$, $\hat{p} = \hat{p}(\theta, \varphi)$

$\hat{p}$: direction of $\vec{p}$

$$\chi_+ = \begin{bmatrix} \cos\theta/2 \\ \sin\theta/2\, e^{i\varphi} \end{bmatrix}, \quad \chi_- = \begin{bmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2}\, e^{i\varphi} \end{bmatrix}. \quad \begin{cases} \chi_+^\dagger\chi_+ = \chi_-^\dagger\chi_- = \mathbb{1} \\ \chi_+^\dagger\chi_- = \chi_-^\dagger\chi_+ = 0 \end{cases}$$

$$\psi_{+,\sigma}(\vec{p}) = \begin{bmatrix} \chi_+ \\ \dfrac{p}{|E_p|+m}\chi_+ \end{bmatrix}, \begin{bmatrix} \chi_- \\ -\dfrac{p}{|E_p|+m}\chi_- \end{bmatrix}, \quad \psi_{-,\sigma}(\vec{p}) = \begin{bmatrix} -\dfrac{p}{|E_p|+m}\chi_+ \\ \chi_+ \end{bmatrix}, \begin{bmatrix} +\dfrac{p\chi_-}{|E_p|+m} \\ \chi_- \end{bmatrix}$$

Prob. 3. <u>Most of you didn't work out this due!</u>

$$i\partial_t \psi = \vec{\alpha}\cdot\vec{p}\,\psi + m\beta\psi \qquad 1)$$

One can show, for $\psi^\dagger = (\psi^*)^T$,

$$-i\partial_t \psi^\dagger = -\frac{\vec{\nabla}}{i}\psi^\dagger\cdot\vec{\alpha} + m\psi^\dagger\cdot\beta \qquad 2)$$

Note it is not meaningful to have $\vec{\alpha}\cdot\vec{\nabla}\psi^\dagger$; it has to be $\nabla\psi^\dagger\cdot\vec{\alpha}$ as $\vec{\alpha}$ is a matrix, $\psi^\dagger$ is a "row" vector.

$$\psi^\dagger \times \text{Eq. 1} - \text{Eq. 2)} \times \psi = i\partial_t[\psi^\dagger\psi] + i\psi^\dagger\vec{\alpha}\cdot\vec{\nabla}\psi + i\nabla\psi^\dagger\cdot\vec{\alpha}\psi = 0$$

or $\partial_t[\psi^\dagger\psi] + \vec{\nabla}\cdot[\psi^\dagger\vec{\alpha}\psi] = 0.$

Eq. 2 derived in this way (Most convenient way) (Set $m=0$)

$\alpha = 1,2,3,4$

$$i\partial_t\psi_\alpha = -i(\alpha\cdot\vec{\nabla})_{\alpha\beta}\psi_\beta, \qquad \alpha\cdot\vec{\nabla} = \alpha_x\nabla_x + \alpha_y\nabla_y + \alpha_z\nabla_z$$

$$-i\partial_t\psi_\alpha^* = +i(\alpha^*\cdot\vec{\nabla})_{\alpha\beta}\psi_\beta^*$$

α = Hermitian

$$= i(\vec{\nabla}\psi_\beta^*)\cdot\alpha^*_{\alpha\beta} = i\vec{\nabla}\psi_\beta^*\,\alpha_{\beta\alpha}$$

equivalently $\quad -i\partial_t\psi^\dagger = i\vec{\nabla}\psi^\dagger\cdot\vec{\alpha} + m\psi^\dagger\cdot\beta$

$$\psi = \begin{bmatrix}\chi\\ \phi\end{bmatrix} \qquad \psi^\dagger = [\chi^\dagger, \phi^\dagger]$$

"column" "Row"

Prob. 4

$$\psi_{+,\sigma}(\bar{p}) = N\begin{bmatrix} \chi_\pm \\ \dfrac{\bar{\sigma}\cdot\vec{p}}{|E_p|+m}\chi_\pm \end{bmatrix}, \quad \psi_{-,\sigma}(\bar{p}) = N\begin{bmatrix} -\dfrac{\bar{\sigma}\cdot\vec{p}}{|E_p|+m}\chi_\pm \\ \chi_\pm \end{bmatrix}$$

Normalization Condition $N = \dfrac{1}{\sqrt{1+\dfrac{p^2}{(|E_p|+m)^2}}} \hat{=} 1$ when $p\to 0$

Keeping $\chi_+^\dagger\chi_+ = \chi_-^\dagger\chi_- = \mathbb{1}$

Current density $j_\alpha = \sum_\sigma \psi^\dagger_{\pm,\sigma}(\bar{p})\,\vec{\alpha}\,\psi_{\pm,\sigma}$

$\alpha = x, y, z$; σ ↓ Summation over Spin states

For $E_p = \pm|E_p|$,

$$\vec{j}_{\pm,\alpha} = \pm\sum_\sigma \chi_\sigma^\dagger \frac{\vec{\sigma}\cdot\vec{p}\,\sigma_\alpha + \sigma_\alpha\,\vec{\sigma}\cdot\vec{p}}{|E_p|+m}\chi_\sigma \cdot N^2$$

$|E_p|+m \cong 2m, \quad N^2 \cong \mathbb{1}. \quad \{\sigma_\alpha, \sigma_\beta\} = 2\delta_{\alpha\beta}$

$$j_{\pm,\alpha} = \pm\frac{1}{2m}\sum_\sigma \chi_\sigma^\dagger \underbrace{(\sigma_\alpha\sigma_\beta + \sigma_\beta\sigma_\alpha)}_{2\delta_{\alpha\beta}}\chi_\sigma\, p_\beta$$

$$= \pm\frac{p_\alpha}{m} + O(p^3)\ldots$$

a) $p\to 0$, $\vec{j}_+ = \dfrac{\vec{p}}{m}$, $\vec{j}_- = -\dfrac{\vec{p}}{m}$ (Opposite Current)

b) No, they don't carry the same Current

$$j_{\vec{p}}(E_p^+) = -j_{\vec{p}}(-E_p^-) \quad \text{(See above)}$$

c) No, it doesn't.

d) $\vec{j}_{\pm} = \pm \frac{\vec{P}}{m} \sum_{\sigma} \chi_{\sigma}^{\dagger} \chi_{\sigma}$, $\chi_{\sigma}^{\dagger} \chi_{\sigma} = \mathbb{1}$, $\chi_{\sigma}^{\dagger} \chi_{\sigma} = \chi_{\sigma}^{\dagger\prime} U^{\dagger} U \chi_{\sigma}^{\prime}$

for positive/Negative energy states $= \chi_{\sigma}^{\dagger\prime} \chi_{\sigma}^{\prime}$

SU(2) Rotation invariant

d) the current density independent the choice of basis.