Solution

Prob. 2. Following our discussions on the pseudo potential, we have

$$T(E) = g + \frac{g}{\Omega} \sum_{\vec{k}'} \frac{1}{E - \frac{\vec{k}'^2}{\frac{1}{2}} + i\delta} T(E)$$

or

$$T(E) = \frac{1}{\frac{1}{g} + \underbrace{\frac{2}{\Omega} \sum_{\vec{k}'} \frac{1}{\vec{k}'^2 - k_E^2 - i\delta}}_{I(E)}}$$

$$k_E^2 = \sqrt{2E}$$

$$I(E) = \frac{2\times}{(2\pi)} \int \frac{d^{(3)}\vec{k}'}{(2\pi)^3} \frac{1}{\vec{k}'^2 - k_E^2 - i\delta} = \frac{1}{\pi^2} \int_0^\Lambda \frac{dk'\, k'^2}{k'^2 - k_E^2 - i\delta}$$

$$= \frac{1}{\pi^2} \left[\int_0^\Lambda dk' \cdot 1 + \int_0^\Lambda dk' \frac{k_E^2}{k'^2 - k_E^2 - i\delta} \right]$$

$$= \underset{O(\Lambda)}{\frac{\Lambda}{\pi^2}} + \underset{O(\Lambda^0)}{\frac{i k_E}{2\pi}} - \underset{O(\Lambda^{-1})}{\frac{k_E^2}{\pi^2 \Lambda}} + O\left(\frac{1}{\Lambda^2}\right) + \cdots \quad (\Lambda \to \infty)$$

Putting together,

$$T(E) = \frac{1}{\frac{1}{g} + \frac{\Lambda}{\pi^2} - \frac{k_E^2}{\pi^2 \Lambda} + \frac{1}{2\pi} i k_E}$$

Note that $\frac{k_E^2}{\pi^2 \Lambda}$ is much less that $\frac{1}{2\pi} k_E$ if $k_E/\Lambda \ll 1$

it gives the leading correction to $\Lambda \to \infty$ limit.

1) According to our discussions,

$$f(\theta) = -\frac{2}{4\pi} T(E) = \frac{\sin\delta \; e^{i\delta}}{k}$$

And $$T(E) = \frac{2\pi}{k} \frac{k a_x}{1 + i k a_x} \quad \text{when } k = k_E \ll \Lambda.$$

$$a_x = \frac{1}{2\pi}\left(\frac{1}{g} + \frac{\Lambda}{\pi^2}\right)^{-1},$$ a_x, a length as a function of (g, Λ).

One can easily verify that

$$\boxed{\tan\delta = -k a_x} \quad \text{for } k \ll \Lambda.$$

Taking into account that $a_s = -\lim_{k\to 0} \frac{\sin\delta}{k} = a_x$

One finds that $$a_s = \frac{1}{2\pi}\left(\frac{1}{g} + \frac{\Lambda}{\pi^2}\right)^{-1}$$

$g \to \infty$, $a_s = \frac{\pi}{2\Lambda} \sim O(R_0)$ Consistent with the hard sphere Result.

In general, $$a_s = \frac{g}{2\pi\left(1 + \frac{g\Lambda}{\pi^2}\right)}, \quad (g > 0)$$

2) $$a_s = \frac{1}{2\pi}\left(\frac{1}{g} + \frac{\Lambda}{\pi^2}\right)^{-1}$$

if $\frac{1}{g} + \frac{\Lambda}{\pi^2} = 0$ or $1/g = -\frac{\Lambda}{\pi^2}$ (Attractive), $a_s \to \infty$

but if Repulsive, $a_s \leq \frac{\pi}{2\Lambda}$

And $\delta \simeq -k a_s = -\frac{k\pi}{2\Lambda}$ if $k \ll \Lambda$.

For repulsive interaction, $|\delta| \approx + ka_s \leq -\frac{k\pi}{\Lambda 2}$

For $k << \Lambda$. The phase shifts are small.

3) For attractive case, $|a_s| \to \infty$ if $|g| = \frac{\pi^2}{\Lambda}$

This defines Resonance. Note that $a_s = \pm\infty$ depending how the resonance is approached.

Fig. 1.

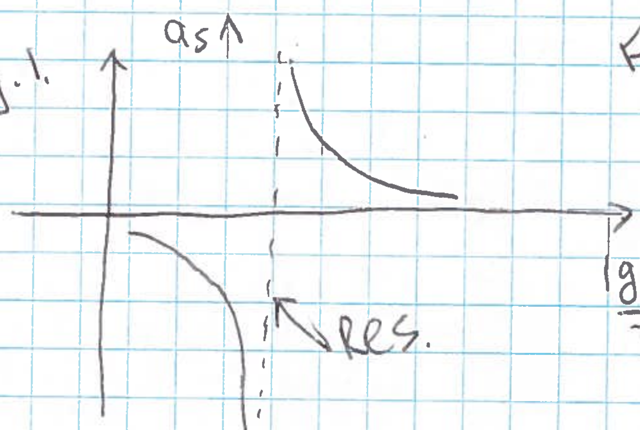
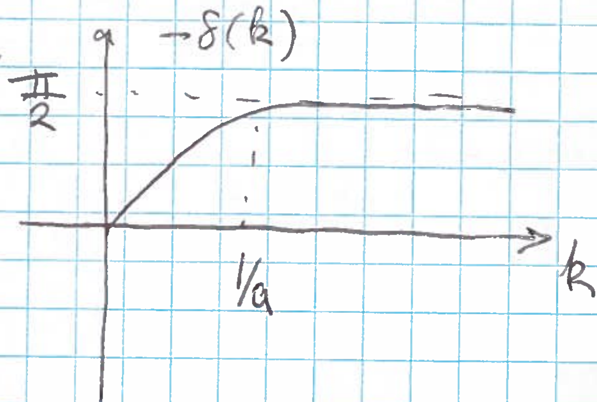


Fig 2



$\delta = -\arctan ka_s$, $\boxed{-\delta \doteq \frac{\pi}{2} - \frac{1}{ka_s}}$ if $ka_s >> 1$

$(k/\Lambda << 1)$

Note $|a_s| >> \frac{1}{\Lambda}$.

4) As shown above, $a_s = -\lim_{k\to 0} \frac{\sin\delta}{k} = \frac{g}{\left(1+\frac{g\Lambda}{\pi^2}\right)2\pi}$ leading order

And this is valid on both side of Resonance when $|a_s| \to \infty$.

(See Fig 1, Fig 2)

$\tan\delta = -ka_s$, $\delta = -\arctan ka_s = -\frac{\pi}{2}\left(1 - \frac{1}{ka_s} \cdots\right)$

Taking into higher order corrections

$$\tan\delta = \frac{-ka_s}{1 + \frac{a_s k^2}{\Lambda\pi^2}(-1)} \qquad \text{Following } T(E).$$

$$\delta(k) = -\arctan\left(\frac{ka_s}{1 - \frac{k^2 a_s}{\Lambda\pi^2}}\right)$$

$$-\delta(k) = \begin{cases} \frac{\pi}{2} + \frac{k\pi^2}{\Lambda}, & (\Lambda a_s) \gg ka_s \gg \left(\frac{\Lambda a_s}{\cancel{}}\right)^{1/2} \\ \frac{\pi}{2} - \frac{1}{ka_s}, & \left(\frac{a_s\Lambda}{\cancel{}}\right)^{1/2} \gg ka_s \gg 1 \end{cases}$$

if $\Lambda \to \infty$ first, $\quad -\delta(k) = \frac{\pi}{2} - \frac{1}{ka_s}$.

5) $\quad T(E) = \dfrac{2\pi a_s}{1 + i a_s k_E}$

take $E = -E_B$, $\quad T(-E_B) = \dfrac{2\pi a_s}{-a_s k_B + 1}$, $\quad k_B = \sqrt{2E_B}$

$T(-E_B)$ has a pole at $E_B = +\frac{1}{2a_s^2} \left(= \frac{1}{2a_B^2}\right)$ definition

so $\quad \boxed{a_B = a_s}$

Prob 1. (Following Textbook page 531)

$$f(\theta) = -\frac{2g}{\mu_0^2 + q^2} \quad , \text{ if } V(r) = g\frac{e^{-\mu_0 r}}{r} \quad (\text{Yukawa form})$$

$$q^2 = 2k^2(1-\cos\theta)$$

expanding in terms of q/μ_0, we have

$$f(\theta) = -\frac{2g}{\mu_0^2} + \frac{2g}{\mu_0^2}\frac{q^2}{\mu_0^2} + \cdots$$

$$\approx \underbrace{-\frac{2g}{\mu_0^2}}_{\substack{\text{S-wave} \\ \text{leading term}}} - \underbrace{\frac{4g}{\mu_0^2}\frac{k^2}{\mu_0^2}}_{\text{P-wave}} \sqrt{4\pi}\, Y_1^{m=0}(\theta) + \cdots$$

Comparing with the expression for phase shifts, one obtains

$$\delta_{l=0}(k) = -k\underbrace{\boxed{\frac{2g}{\mu_0^2}}}_{\substack{\text{Scattering} \\ \text{length}}} \;, \qquad \delta_{l=1}(k) = -k^3 \underbrace{\boxed{\frac{4g}{\mu_0^2}\cdot\frac{\sqrt{3}}{\mu_0^2}}}_{\text{Scattering vol.}}$$

[Note this is obtained in the Born approximation]