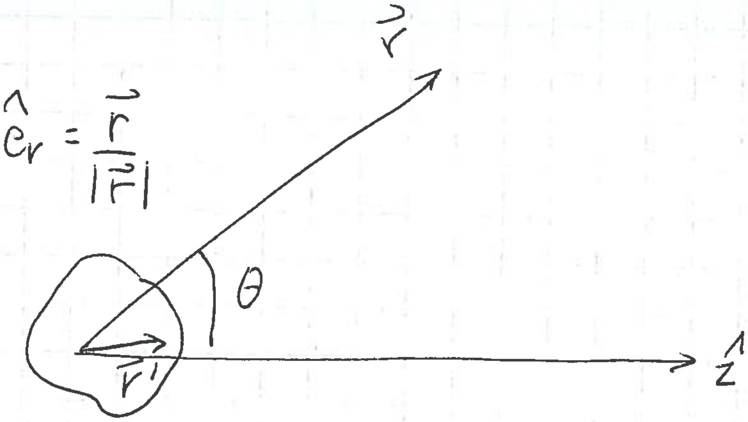
Home work Set IV

$$\begin{cases} \vec{k}_f = |\vec{k}_i|\,\hat{e}_r, & \hat{e}_r = \dfrac{\vec{r}}{|\vec{r}|} \\ \vec{k}_i = |\vec{k}_i|\,\hat{e}_z, & \\ \hat{e}_r \cdot \hat{e}_z = \cos\theta & \end{cases}$$



$$f(\theta,\varphi) = f(\vec{k}_f\cdot\vec{k}_i) = -\frac{2}{4\pi}\int d^3\vec{r}\,'\, e^{-i\vec{k}_f\cdot\vec{r}\,'}\, V(\vec{r}\,')\,\psi^{TOT}_{\vec{k}_i}(\vec{r}\,')$$

(Note $\psi^{TOT}_{\vec{k}_i}(\vec{r}\,')$ is the solution when the incident wave is written as $1\cdot e^{i\vec{k}_i\cdot\vec{r}}$)

*** Recall that $f(\theta,\varphi) = \sum_l \sqrt{4\pi(2l+1)}\ \frac{\sin\delta_l}{k}\, e^{i\delta_l}\, Y_l^m(\theta)$ — phase shifts

$$= \sum_l \sqrt{4\pi(2l+1)}\ a_l\, Y_l^0(\theta)$$

$\to$ partial wave scattering amplitude.

In the Born Approximation (linear in $V(\vec{r}\,')$)

$$f(\theta,\varphi) \approx -\frac{2}{4\pi}\int d^3\vec{r}\,'\ V(\vec{r}\,')\, e^{i(\vec{k}_f-\vec{k}_i)\cdot\vec{r}\,'}$$

$\vec{q}$ = Change of Momentum

Prob. 1 (1)

$$f(\theta,\varphi) = -\frac{2}{4\pi}\int dr\, r^2 \int d\Omega\ (-V_0\,\theta(r_0-r)\, e^{i\vec{q}\cdot\vec{r}})$$

$$= -\frac{2V_0}{q}\int_0^{r_0} dr\ r\,\sin qr = -\frac{2V_0}{q^3}\left(\sin qr_0 - qr_0\cos qr_0\right)$$

$$\frac{d\sigma}{d\Omega} = |f(\theta,\varphi)|^2 \longrightarrow \text{conclusion}$$

(2) expansion of qr_0 leading to (in the Born Approximation)

$$f(\theta,\varphi) \overset{qr_0\to 0}{=} +2V_0 r_0^3 \left[\left(\frac{1}{2!}-\frac{1}{3!}\right) \overset{(-)}{+} \left(\frac{1}{4!}-\frac{1}{5!}\right) q^2 r_0^2 + \cdots \right]$$

Note $q^2 r_0^2 = 2k^2 r_0^2 (1-\cos\theta)$, so

$$f(\theta,\varphi) = +2V_0 r_0^3 \left(\frac{2}{6} - \frac{1}{20}\cdot 2\cdot k^2 r_0^2 (1-\cos\theta) \right)$$

$$= +2V_0 r_0^3 \left(\frac{2}{6} - \frac{1}{30} k^2 r_0^2 + \cdots \right) \leftarrow \text{isotropic}$$

$$+2V_0 r_0^3 \left(\frac{1}{30} k^2 r_0^2 \cdot \sqrt{4\pi}\, Y_1^0(\theta) \right) \leftarrow p\text{-wave}$$

S-wave $\delta_{\ell=0} \doteq +k \cdot \underbrace{\frac{2}{3} V_0 r_0^3}_{a_s}$, $(r_0 \to 0)$ $\quad \delta_{\ell=1} = +\left(+\underbrace{\frac{\sqrt{4\pi}}{15} V_0 r_0^2 \cdot r_0^3}\right) k^3$

a_s — Scattering length $\quad$ — Scattering vol.

(Notice the signs)

Prob 2.

$$f(\theta,\varphi) = -\frac{2}{4\pi} \int d^{(3)}\vec{r}\; V_0 e^{-r^2/r_0^2}\, e^{-i\vec{q}\cdot\vec{r}}$$

$$= -\frac{V_0}{iq} \int_0^\infty dr\, r \left[e^{-r^2/r_0^2 + iqr} - \text{c.c.} \right] \leftarrow \text{After } d\Omega \text{ integration}$$

$$= +\frac{V_0}{iq} \times \frac{r_0^2}{2} \int_{-\infty}^{+\infty} d\tilde{r} \left[e^{-(\tilde{r} + iq\frac{r_0}{2})^2 + (-1)\frac{q^2 r_0^2}{4}} \cdot \tilde{r} - e^{-(\tilde{r} - iq\frac{r_0}{2})^2 - \frac{q^2 r_0^2}{4}} \cdot \tilde{r} \right]$$

$$= (-)\frac{V_0 r_0^2}{2iq} e^{-\frac{q^2 r_0^2}{4}} (iqr_0) \int_{-\infty}^{\infty} d\tilde{r}\, \underbrace{e^{-\tilde{r}^2}}_{\sqrt{\pi}} = \boxed{\frac{V_0 r_0^3}{2} e^{-\frac{q^2 r_0^2}{4}} \sqrt{\pi}}$$

(−)

$\longrightarrow$ Conclusion