Solutions

Prob. 1

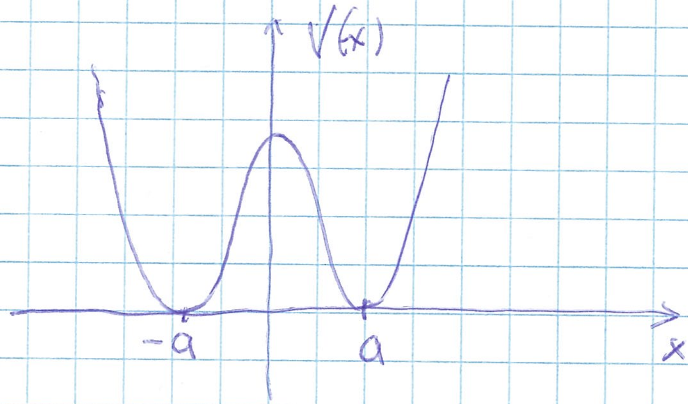
$$S_E(\{x(\tau)\}) = \int_{-\tau/2}^{\tau/2} \mathcal{L}_E(\{x(\tau')\})\, d\tau' , \qquad \mathcal{L}_E = \frac{\dot{x}^2}{2} + V(x)$$

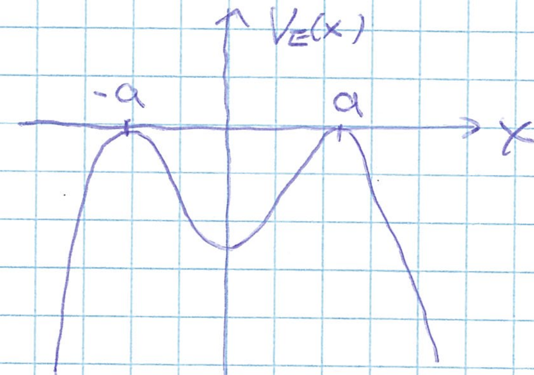
Equation of Motion for classical Solution is the same as $\mathcal{L}$. Namely,

$$\frac{d}{dt}\frac{\partial \mathcal{L}_E}{\partial \dot{x}} - \frac{\partial \mathcal{L}_E}{\partial x} = 0$$

However, $V_E(x) = -V(x)$ so that $\mathcal{L}_E = \frac{\dot{x}^2}{2} - V_E(x)$

$$\boxed{\ddot{x} = -\frac{\partial V_E(x)}{\partial x} = +\frac{\partial V(x)}{\partial x}}$$





The solution to our problem with $x(\frac{\tau_\infty}{2}) = a$, $x(-\frac{\tau_\infty}{2}) = -a$ shall have $E = 0$.

$$\int_{\tau_0}^{\tau} d\tau = \int_0^{x(\tau)} \frac{dx'}{\sqrt{2V(x')}} \longrightarrow \tau = \tau_0 + \int_0^{x} \frac{dx'}{\sqrt{2V(x')}} \quad (x(\tau_0) = 0)$$

$$V(x) = \frac{\omega^2}{8}\frac{1}{a^2}(x^2 - a^2)^2$$

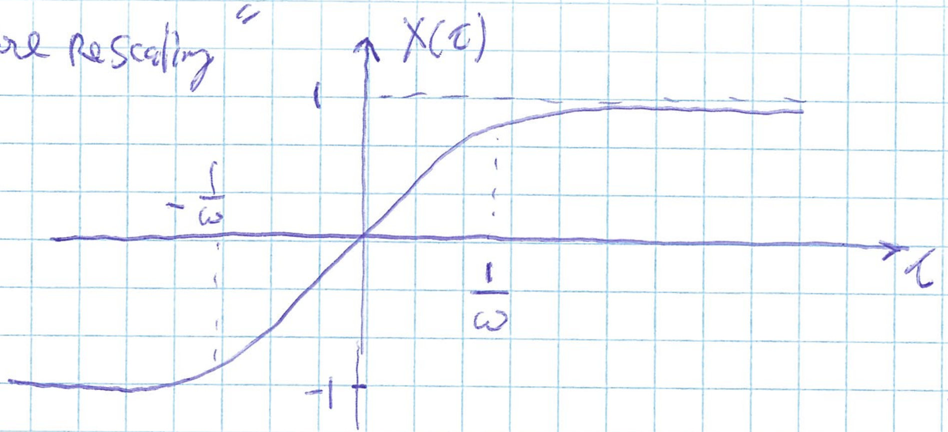
Set $a = \omega = 1$

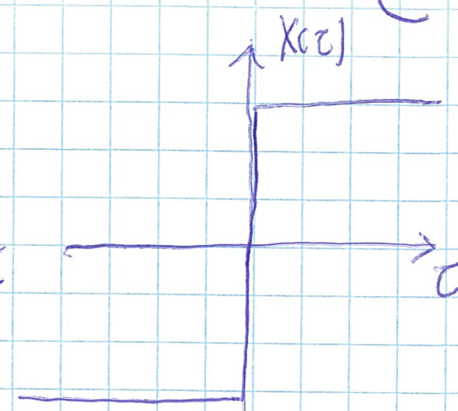
(which means $x \to \frac{x}{a}$, $\tau \to \tau\omega$)

$$\boxed{\tau = \tau_0 + \log\frac{1+x}{1-x}} \qquad x \in [-1, 1]$$

or $$x = \frac{1 - e^{-(\tau-\tau_0)}}{1 + e^{-(\tau-\tau_0)}}$$

"before rescaling"





$\omega \to \infty$

Note that 1) $X \to 1 - 2e^{-(\tau-\tau_0)}$ $\quad \tau - \tau_0 \to +\infty$

2) $X \to -1 - 2e^{+(\tau-\tau_0)}$ $\quad \tau - \tau_0 \to -\infty$

This is Robust.

Consider a general double-well potential. when

$$V(x) \underset{x \to \pm a}{=} \frac{\omega^2}{2}(x \pm a)^2 + O\left((x \pm a)^3 + \cdots\right)$$

$$\stackrel{\circ}{=} \frac{1}{2}(x \pm 1)^2$$

$$I = \int_0^X \frac{dx'}{\sqrt{2V(x')}} \underset{x \to \pm 1}{=} \underbrace{\int_A^X \frac{dx'}{\sqrt{2V(x')}}}_{\text{Singular part}} + \underbrace{\int_A^0 \frac{dx'}{\sqrt{2V(x')}}}_{\text{Regular part}}$$

$$\stackrel{\circ}{=} \log\frac{1+X}{1-X} - \log\frac{1+A}{1-A} + f(A)$$

→ Regular

When $X \to 1$, $\quad I \to \log\frac{1+X}{1-X}$

Therefore $\quad \tau = \tilde{\tau}_0(A) + \log\frac{1+X}{1-X}$, $\quad \tilde{\tau}_0(A) = \tau_0 - \log\frac{1+A}{1-A} + f(A)$

$$\boxed{X \stackrel{\tau \to \infty}{=} \frac{1 - e^{-(\tau - \tilde{\tau}_0(A))}}{1 + e^{-(\tau - \tilde{\tau}_0(A))}}}$$

⟸ Universal structure

Prob. 2.

$$S_{E\,cl}[x(t)] = \int_{-\frac{\tau_\infty}{2}}^{+\frac{\tau_\infty}{2}} \left[\frac{\dot{x}_{cl}^2}{2} + V(x) \right] dt$$

$$= \int_{-\frac{\tau_\infty}{2}}^{\frac{\tau_\infty}{2}} \dot{x}_{cl}^2 \, dt, \qquad x_{cl} = \frac{1-e^{-\tau}}{1+e^{-\tau}} \qquad (\tau_0 = 0)$$

Restoring the units

$$= \int_{-\frac{\tau_\infty}{2}}^{\frac{\tau_\infty}{2}} \frac{2e^{-\tau} d\tau}{(1+e^{-\tau})^4} = \frac{2}{3} \Leftarrow \frac{2}{3}\omega a^2 = \boxed{\frac{2\omega a^2}{3}}$$

Note

$$S_E = \int_{-\frac{\tau_\infty}{2}}^{\frac{\tau_\infty}{2}} \dot{x}_{cl}^2 \, dt = \int_{-1}^{+1} \sqrt{2V(x)} \, dx = \frac{2}{3}(\omega a^2)$$

unit for action

Prob. 3. One has to identify which parameter is appropriate here

$$\left\langle \frac{1}{\tau_\infty} \int y^2(t) \, dt \right\rangle_{\text{all paths average}} = \frac{1}{\tau_\infty} \int_{-\frac{\tau_\infty}{2}}^{\frac{\tau_\infty}{2}} \langle y^2(t) \rangle_{\text{all paths}} dt = C_Q$$

C_Q : strength of quantum fluctuations.

This measures the average quantum fluctuations over time t_∞. "And let's fix τ_0 in this case."

In the Gaussian Approximation, (C_n - Real quantity)

$$y(t) \doteq \sum_{n=1}^{\infty} Y_n(t) \, C_n, \qquad Y_n(t) = \sqrt{\frac{2}{\tau_\infty}} \sin \frac{2n\pi}{2\tau_\infty} t,$$

$$C_Q = \frac{1}{\tau_\infty} \int_{-\frac{\tau_\infty}{2}}^{\frac{\tau_\infty}{2}} \sum_{n,n'} \langle C_n C_{n'} \rangle \cdot Y_n(t) Y_{n'}(t) \, dt$$

$$= \frac{1}{\tau_\infty} \sum_n \langle C_n C_n \rangle$$

All possible paths

$$\mathcal{U}\left(\frac{\tau_\infty}{2}, -\frac{\tau_\infty}{2}\right) = \underbrace{\int \mathcal{D}C_n \, e^{-\frac{1}{2}\sum_n C_n^2(\omega_n^2+\omega^2)}}_{\text{Gaussian fluctuations}} \cdot \underbrace{e^{-S_E(\{x_{cl}\})}}_{\text{instanton ..}}$$

$$\langle C_n C_n \rangle = \frac{\int \mathcal{D}C_n \, e^{-\frac{1}{2}\sum_n C_n^2(\omega_n^2+\omega^2)} \, C_n^2}{\int \mathcal{D}C_n \, e^{-\frac{1}{2}\sum_n C_n^2(\omega_n^2+\omega^2)}} = \frac{1}{\omega_n^2+\omega^2}$$

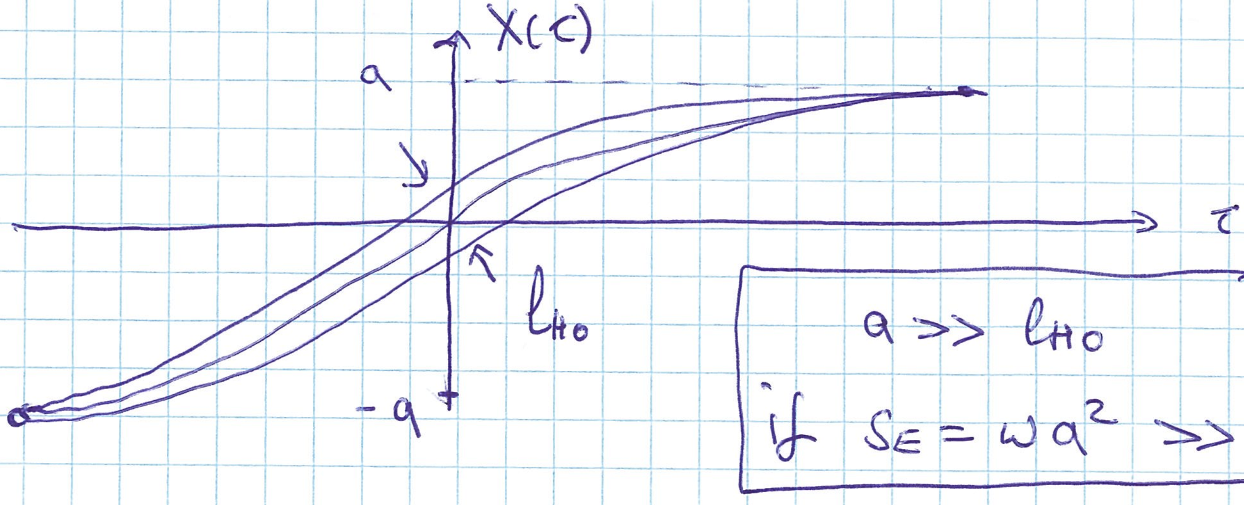
$$C_Q = \frac{1}{\tau_\infty}\sum_n \langle C_n^2 \rangle = \frac{1}{\tau_\infty}\sum_{n=0}^{\infty} \frac{1}{\omega_n^2+\omega^2}$$

$$= \frac{1}{\tau_\infty}\left[\frac{1}{2}\sum_{-\infty}^{\infty}\frac{1}{\omega_n^2+\omega^2} - \frac{1}{2\omega^2}\right],$$

$$\overset{\tau_\infty\to\infty}{=} \left[\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{dz}{z^2+\omega^2} - \underbrace{\frac{1}{2\tau_\infty\omega^2}}_{\to 0}\right] = \frac{1}{2}\cdot\frac{1}{\omega}$$

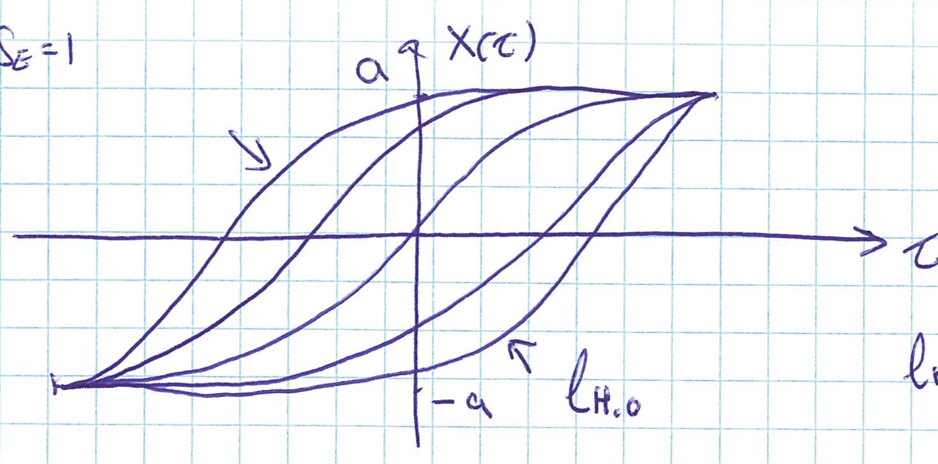
Remember for a Harmonic oscillator, for ground state

$$\langle X^2 \rangle_{H.O.} = \frac{1}{2\omega} = l_{H.O}^2, \qquad l_{H.O.} \text{ denote harmonic length}$$



$$\boxed{\begin{array}{c} a \gg l_{HO} \\ \text{if } S_E = \omega a^2 \gg 1 \end{array}}$$

but if $S_E = 1$

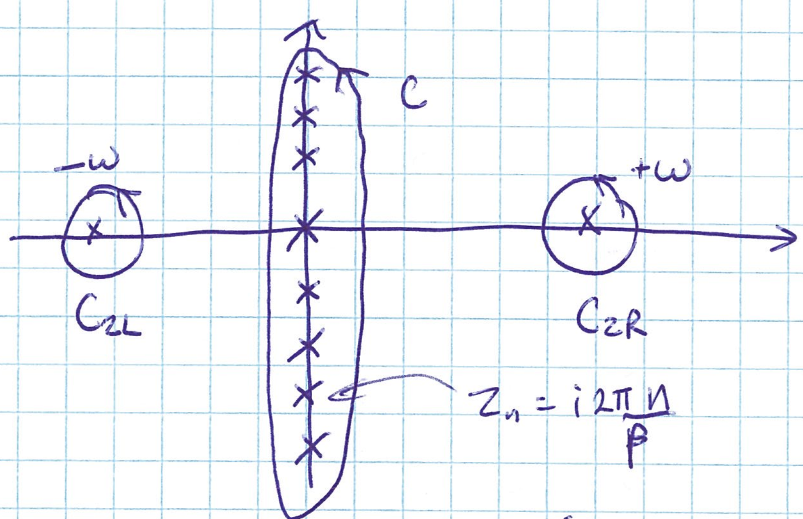


$l_{H0} \sim a$

Note

The sum can be computed using the analytic continuation trick discussed before. τ_∞ is sent to ∞.

$$C_Q = \frac{1}{\tau_\infty}\left[\frac{1}{2}\sum_{n=-\infty}^{\infty}\frac{1}{\omega_n^2+\omega^2} - \frac{1}{2\omega^2}\right]$$



Poles are shown explicitly. Rest are analytic region where contour C, C_{2R}, C_{2L} can be deformed into.

definite $2\tau_\infty = \beta = \frac{1}{T_1}$ (effective temperature)

$$C_Q = \beta^{-1}\left[\sum_{n=-\infty}^{+\infty}\frac{1}{\omega_n^2+\omega^2} - \frac{1}{\omega^2}\right]$$

$$\sum_{n=\infty}^{+\infty}\frac{1}{\omega_n^2+\omega^2} = \frac{\beta}{(2\pi i)}\oint_C dz\,\frac{1}{\omega^2-z^2}\,\frac{1}{e^{\beta z}-1}$$

$$= -\frac{\beta}{(2\pi i)}\left(\oint_{C_{2R}} + \oint_{C_{2L}}\right) dz\,\frac{1}{\omega^2-z^2}\,\frac{1}{e^{\beta z}-1}$$

$$= \frac{\beta}{2\omega}\,\frac{e^{\omega\beta}+1}{e^{\omega\beta}-1} = \frac{\beta}{2\omega}\left(2n_B+1\right)$$

$$n_B = \frac{1}{e^{\omega\beta}-1}$$ Planck-distribution. $\beta = 2\tau_\infty = \frac{1}{T}$

$$C_Q = \frac{1}{\omega}\left(n_B + \frac{1}{2}\right) - \frac{1}{\omega^2\beta}$$

$$= \boxed{\frac{1}{\omega}\left[n_B + \frac{1}{2} - \frac{1}{\omega\beta}\right]}$$ if τ_∞ is finite

One of the famous Result in instanton calculations.

- n_B : "thermal" occupation of Harmonic states
- $\frac{1}{2}$: Quantum zero pt motion
- $\frac{1}{\omega\beta}$: is there to remove IR divergence

Note that $$C_Q = \frac{1}{2\omega} + \frac{1}{\omega}\underbrace{\left(n_B - \frac{1}{\omega\beta}\right)}_{\text{well behaved as } \omega\to 0}$$

$\to 0$ as $\beta \to \infty$.