Solution to HW Set I (Jan 16, 2017)

1) $\psi_{TOT} = h^{(2)}(kr) + c\, h^{(1)}(kr)$ (up to a normalization factor)

Taking into account $h^{(2)} = h^{(1)*}$ and

$$h^{(1)*} \nabla h^{(2)} = (h^{(2)*} \nabla h^{(1)})^*$$

$$\vec{J}_{TOT} = \underbrace{\frac{1}{2i}\left(h^{(2)*}\nabla h^{(2)} - h^{(2)}\nabla h^{(2)*}\right)}_{\vec{j}_1} + \frac{cc^*}{2i}\underbrace{\left(h^{(1)*}\nabla h^{(1)} - h^{(1)}\nabla h^{(1)*}\right)}_{\vec{j}_2}$$

Using the asymptotic property of $h^{(1)}$, $h^{(2)}$, we find that

$$j_r = \vec{J}_{TOT} \cdot \hat{e}_r = \frac{k}{r^2}\left(|c|^2 - 1\right).$$

Since the divergence of the current is

$$\vec{\nabla}\cdot\vec{j} \overset{\vec{r}\to\infty}{=\!=} \underbrace{4\pi k\left(|c|^2-1\right)}_{\text{Gauss' Law}} \delta(\vec{r})$$

$|c|^2 = 1$ so that $\frac{\partial \rho}{\partial t} = 0$ and probability current is conserved.

2) Following the S-wave equation, one finds

$$\tan(kR_0 + \delta_0(k)) = \frac{k}{\tilde{k}} \tan(\tilde{k} R_0)$$

$k = \sqrt{2E}$, $\tilde{k} = \sqrt{k^2 + k_0^2}$, $k_0 = \sqrt{2V_0}$, V_0 – depth.

or $\delta_0(k) = \arctan\left(\frac{k}{\tilde{k}} \tan(\tilde{k} R_0) - kR_0\right)$

3) $B = (k_0 R_0)^2 \ll 1$, $kR_0 \ll 1$

$$\delta_0(k) = \frac{1}{6} k R_0 (k_0 R_0)^2 + O(k^3 R_0^3)$$

$$\overset{k\to 0}{=} -k a_s \qquad \underset{\text{scattering length}}{a_s} = -\frac{1}{6} R_0 (k_0 R_0)^2$$

Or $\frac{a_s}{R_0} = -\frac{1}{6} B$, B – Bohn parameter.

Generally, $a_s = -\lim_{k\to 0} \frac{\tan\delta(k)}{k}$

characterizing the linear k dependence of $\delta(k)$ near $k=0$.

4) Taking the definition of a_s, $\tan\delta(k) = -ka_s$ as $k\to 0$.

$$\sin\delta(k) = \frac{-ka_s}{\sqrt{1+(ka_s)^2}}$$

$$\underset{\text{S-wave only}}{\sigma(k\to 0)} = 4\pi \frac{\sin^2\delta(k)}{k^2} = 4\pi \frac{a_s^2}{(1+k^2 a_s^2)} \to 4\pi a_s^2$$

$\sigma \to \infty$ when $a_s \to \infty$.

For squre well prob, $a_s = -\left(\frac{\tan(\tilde{k}R_0)}{\tilde{k}} - R_0\right)$

$a_s \to \infty$ Requires $\tan(\tilde{k}R_0) = \infty$ when $\tilde{k} \to k_0$.

$$\tilde{k}R_0 = k_0 R_0 = (n+\tfrac{1}{2})\pi, \qquad \boxed{k_0 = \frac{(n+\frac{1}{2})\pi}{R_0}}$$

Resonance

5) phase shifts are determined by

$$\tilde{k}\,\frac{j_0'(\tilde{\rho})}{j_0(\tilde{\rho})}\bigg|_{\tilde{\rho}=\tilde{\rho}_0} = k\,\frac{j_0'(\rho)\cos\delta_0 - n_0'(\rho)\sin\delta_0}{j_0(\rho)\cos\delta_0 - n_0(\rho)\sin\delta_0}\bigg|_{\rho=\rho_0}$$

$\tilde{\rho}_0 = \tilde{k}R_0,\quad \rho_0 = kR_0$

or $$\frac{\tilde{k}}{\tan\tilde{k}R_0} = \frac{k}{\tan(kR_0+\delta)} \qquad \text{Eq. 1}$$

bound state can be formulated in a similar way.

For S-wave, $\boxed{\psi_B \sim \frac{e^{-r/a_B}}{r}.}$

$$\frac{\tilde{k}_B}{\tan\tilde{k}_B R_0} = -\frac{1}{a_B} \qquad \text{Eq. 2}$$

Notice that $\tilde{k} \xrightarrow{k\to 0} k_0,\quad \tilde{k}_B = \sqrt{-\left(\frac{1}{a_B}\right)^2 + k_0^2} \xrightarrow{a_B\to\infty} k_0$

(Eq. 1) LHS at $k\to 0$ = (Eq. 2) LHS at $a_B\to\infty$

or $$\lim_{k\to 0}\frac{\tan(kR_0+\delta)}{k} = \lim_{a_B\to\infty} a_B$$

$$= \lim_{k\to 0}\frac{\tan\delta(k)}{k} = -\frac{1}{a_s} \quad \text{definition}$$

Therefore $\frac{1}{a_s} = \frac{1}{a_B}$ or $a_s = a_B$ near Reso.

(in limit of $a_B \gg R_0$)

$$\boxed{\text{Resonace takes place every time when a bound state approaches } E_B = -\frac{1}{2a_B^2} \to 0. \quad **}$$

6) $a_B \gg r_0$, $kr_0 \lesssim 1$, one can express

$$\tan\delta(k) = -ka_s = -ka_B$$

$$\sigma = \frac{4\pi a_B^2}{1 + k^2 a_B^2} \quad \text{for low energy scattering near Resonance.}$$

7) lth partial wave phase shifts $\delta_l(k)$ given by

$$\boxed{\tilde{k}\,\frac{j_l'(\tilde{\rho})}{j_l(\tilde{\rho})}\bigg|_{\tilde{\rho}=\tilde{\rho}_0} = k\,\frac{j_l'(\rho)\cos\delta_l(k) - n_l'(\rho)\sin\delta_l(k)}{j_l(\rho)\cos\delta_l(k) - n_l(\rho)\sin\delta_l(k)}\bigg|_{\rho=\rho_0}}$$

Note $\rho_0 \ll 1$ but $\tilde{\rho}_0$ can be large.

$$j_l(\rho) \overset{\rho\to 0}{=} \frac{\rho^l}{(2l-1)!!}, \quad n_l(\rho) \overset{\rho\to 0}{=} -\frac{(2l-1)!!}{\rho^{l+1}}$$

$$\tan\delta_l(k) = \frac{\rho_0^{(2l+1)}}{(2l-1)!!(2l+1)!!}\,\frac{l}{l+1}\left[\frac{1 - \frac{1}{l\cancel{(2l-1)!!}}\,\frac{\tilde{\rho}_0 j_l'(\tilde{\rho}_0)}{j_l(\tilde{\rho}_0)}}{1 + \frac{1}{(l+1)\cancel{(2l-1)!!}}\,\frac{\tilde{\rho}_0 j_l'(\tilde{\rho}_0)}{j_l(\tilde{\rho}_0)}}\right]$$

$$\boxed{\begin{array}{l} \tilde{\rho}_0 \approx k_0 R_0 \text{ if } kR_0 \ll 1 \\ \text{Near Resonance} \quad \tilde{\rho}_0 \sim O(1). \end{array}}$$

lth Order Resonance Condition:

$$1 + \frac{1}{(l+1)(2l\cancel{-1})!!}\,\frac{\tilde{\rho}_0\, j_l'(\tilde{\rho}_0)}{j_l(\tilde{\rho}_0)}\Bigg|_{k\to 0} = 0$$

For $l=0$, this corresponding to $\tilde{\rho}_0 = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

Generically, if the resonance conditions are Not satifield,

$$\delta_l(k) \sim \frac{\rho_0^{(2l+1)}}{(2l-1)!!\,(2l+1)!!}\, F_l(\tilde{\rho}_0), \qquad F_l(\tilde{\rho}_0) = [\;\cdot\;\ldots\;]$$

l th partial wave is suppressed by $\rho_0^{2l+1} = (kR_0)^{2l+1}$ as $k \to 0$.

8) $k = \frac{2\pi}{\lambda}$, $\lambda \sim 400\,nm$

$R_0 \sim 0.5\,nm$, $\quad kR_0 \sim 10^{-2}$

$$\delta_{l=1}/\delta_{l=0} \sim \rho_0^{2l} = (kR_0)^2 \sim 10^{-4}$$

For ultra cold gases, S-waves are dominating unless (if near $l=1$ Resonance.)