Phys 501; Quantum Mechanics II

Homework Set I (Due 1230pm, Wed, Jan 23 2019)

Objectives:
Understand resonance scattering and the role of the range:

1) S-wave resonance scattering for a short range potential when the range becomes zero. The relation between bound state physics and scattering lengths a_s.
2) S-wave phase shifts and the role of small but finite range physics (in this HW set);
3) P-wave scattering volume (an analogue of scattering length for S-wave), characterizing the phase shifts in the limit of low energy.
4) P-wave scattering: scattering volume and bound state physics
4) Differences and similarities between p-wave and s-wave resonances.
(con't next week)

Math requirements:
** Deriving the asymptotic behaviors of Bessel functions at large or small distances.*

Prob. 1 S-wave Scattering by a short range potential near resonance

The phase shift as a function of k is a linear function of k for small k. The proportionality defines the usual concept of scattering length a_s which can be either positive or negative. Previous homework problem suggests that a_s is smaller than the potential range when the Born parameter is small but can exceed the range when the potential is deep enough.

In this problem, we focus on the limit when a_s is much longer than the range or beyond the Born limit and near resonance. During our lecture, we show that the phase shift is a linear function of scattering length a_s which is equal to the size of bound state a_b in the limit of zero range. This is the universal relation for any long wave length scattering in the presence of short range potentials. In this problem, you can further examine the effect of the range of potential.

1) What is the phase shift as a function of k for scatterings when the wave length is much longer than the range of the potential (and a_s is much larger than the range)? Keep the most dominating contributions due to the finite range. Note that k a_s can be very large as we are near resonance.
2) Within which window of k the phase shift can still be described by the universal relation derived for the zero range potential and within which k-window the range effect becomes **substantial** ? Here again look for the deviation from the zero range limit explicitly and find when the leading order correction becomes important.

Prob. 2 P-wave physics I: weak scattering and bound states (via square well potential)

1) As a preparation, show the four asymptotic relations for an arbitrary "l" in the supplementary materials for HW set 1 using the recursion relation and generating method we discussed.
2) In the limit of weak scattering, or when the Born parameter is small, compute the scattering volume for p-wave scattering. **Compare with the s-wave scattering phase shift in the same limit**. (use the square well potential introduced before).
3) Also in this limit, find out the k dependence of the phase shift for the long wavelength scattering.
4) Find out the critical value of the potential depth when the bound state first appears, with zero binding energy. Compare with the S-wave result explicitly.
5) Show that at the point, the scattering volume becomes infinite.
6) (**For Bonus points**) Find out the scaling relation between the small deviation of the potential depth from the critical value in 4) and the bound state spatial size near resonance where the bound state is much bigger than the range. Hint: Focus on the critical exponent or how the binding energy scales as the potential depth deviates from the critical one or resonance one. Compare with the S-wave result explicitly.

Prob.3. (**For Bonus points**)

For lth partial wave scattering (via square well potential)

1) Find out the condition for resonance (in terms of an algebraic equation for the depth and range, likely involved the spherical Bessel functions.) You shall be able to verify for l=0 and l=1, they correspond to the condition you have found in the previous problems.
2) Show that when this condition is satisfied, the bound state spatial size is infinite.
3) What is the general scaling relation between the bound state size for lth order waves and the potential depth near resonance i.e. the small deviation from the critical value in 2)?