Phys 402: Applications of Quantum Mechanics

Homework 10 (Total 2 problems; due 930am, Thursday, March 31, 2016)
[To receive full credits, please show all necessary steps that lead to your answers.]

Read section 9.1.3 on sinusoidal perturbations.

1) Prob. 9.15 (part (e) optional)

2) Prob. 9.18.

Problem 9.15

(a)

$$\Psi(t) = \sum c_n(t) e^{-iE_n t/\hbar}\psi_n. \quad H\Psi = i\hbar\frac{\partial\Psi}{\partial t}; \quad H = H_0 + H'(t); \quad H_0\psi_n = E_n\psi_n. \quad \text{So}$$

$$\sum c_n e^{-iE_n t/\hbar} E_n\psi_n + \sum c_n e^{-iE_n t/\hbar} H'\psi_n = i\hbar\sum \dot{c}_n e^{-iE_n t/\hbar}\psi_n + i\hbar\left(-\frac{i}{\hbar}\right)\sum c_n E_n e^{-iE_n t/\hbar}\psi_n.$$

The first and last terms cancel, so

$$\sum c_n e^{-iE_n t/\hbar} H'\psi_n = i\hbar\sum \dot{c}_n e^{-iE_n t/\hbar}\psi_n. \quad \text{Take the inner product with } \psi_m:$$

$$\sum c_n e^{-iE_n t/\hbar}\langle\psi_m|H'|\psi_n\rangle = i\hbar\sum \dot{c}_n e^{-iE_n t/\hbar}\langle\psi_m|\psi_n\rangle.$$

Assume orthonormality of the unperturbed states, $\langle\psi_m|\psi_n\rangle = \delta_{mn}$, and define $H'_{mn} \equiv \langle\psi_m|H'|\psi_n\rangle$.

$$\sum c_n e^{-iE_n t/\hbar} H'_{mn} = i\hbar\dot{c}_m e^{-iE_m t/\hbar}, \quad \text{or} \quad \boxed{\dot{c}_m = -\frac{i}{\hbar}\sum_n c_n H'_{mn} e^{i(E_m - E_n)t/\hbar}.}$$

(b) Zeroth order: $c_N(t) = 1$, $c_m(t) = 0$ for $m \neq N$. Then in first order:

$$\dot{c}_N = -\frac{i}{\hbar}H'_{NN}, \quad \text{or} \quad \boxed{c_N(t) = 1 - \frac{i}{\hbar}\int_0^t H'_{NN}(t')\,dt',} \quad \text{whereas for } m \neq N:$$

$$\dot{c}_m = -\frac{i}{\hbar}H'_{mN}e^{i(E_m - E_N)t/\hbar}, \quad \text{or} \quad \boxed{c_m(t) = -\frac{i}{\hbar}\int_0^t H'_{mN}(t')e^{i(E_m - E_N)t'/\hbar}\,dt'.}$$

(c)

$$c_M(t) = -\frac{i}{\hbar}H'_{MN}\int_0^t e^{i(E_M - E_N)t'/\hbar}\,dt' = -\frac{i}{\hbar}H'_{MN}\left[\frac{e^{i(E_M - E_N)t'/\hbar}}{i(E_M - E_N)/\hbar}\right]\Bigg|_0^t = -H'_{MN}\left[\frac{e^{i(E_M - E_N)t/\hbar} - 1}{E_M - E_N}\right]$$

$$= -\frac{H'_{MN}}{(E_M - E_N)}e^{i(E_M - E_N)t/2\hbar}\,2i\sin\left(\frac{E_M - E_N}{2\hbar}t\right).$$

$$P_{N\to M} = |c_M|^2 = \boxed{\frac{4|H'_{MN}|^2}{(E_M - E_N)^2}\sin^2\left(\frac{E_M - E_N}{2\hbar}t\right).}$$

(d)

$$c_M(t) = -\frac{i}{\hbar}V_{MN}\frac{1}{2}\int_0^t \left(e^{i\omega t'} + e^{-i\omega t'}\right)e^{i(E_M - E_N)t'/\hbar}\,dt'$$

$$= -\frac{iV_{MN}}{2\hbar}\left[\frac{e^{i(\hbar\omega + E_M - E_N)t'/\hbar}}{i(\hbar\omega + E_M - E_N)/\hbar} + \frac{e^{i(-\hbar\omega + E_M - E_N)t'/\hbar}}{i(-\hbar\omega + E_M - E_N)/\hbar}\right]\Bigg|_0^t.$$

If $E_M > E_N$, the second term dominates, and transitions occur only for $\omega \approx (E_M - E_N)/\hbar$:

$$c_M(t) \approx -\frac{iV_{MN}}{2\hbar}\frac{1}{(i/\hbar)(E_M - E_N - \hbar\omega)}e^{i(E_M - E_N - \hbar\omega)t/2\hbar}\,2i\sin\left(\frac{E_M - E_N - \hbar\omega}{2\hbar}t\right), \text{ so}$$

$$P_{N\to M} = |c_M|^2 = \frac{|V_{MN}|^2}{(E_M - E_N - \hbar\omega)^2}\sin^2\left(\frac{E_M - E_N - \hbar\omega}{2\hbar}t\right).$$

If $E_M < E_N$ the first term dominates, and transitions occur only for $\omega \approx (E_N - E_M)/\hbar$:

$$c_M(t) \approx -\frac{iV_{MN}}{2\hbar}\frac{1}{(i/\hbar)(E_M - E_N + \hbar\omega)}e^{i(E_M - E_N + \hbar\omega)t/2\hbar}\,2i\sin\left(\frac{E_M - E_N + \hbar\omega}{2\hbar}t\right), \text{ and hence}$$

$$P_{N\to M} = \frac{|V_{MN}|^2}{(E_M - E_N + \hbar\omega)^2}\sin^2\left(\frac{E_M - E_N + \hbar\omega}{2\hbar}t\right).$$

Combining the two results, we conclude that transitions occur to states with energy $E_M \approx E_N \pm \hbar\omega$, and

$$\boxed{P_{N\to M} = \frac{|V_{MN}|^2}{(E_M - E_N \pm \hbar\omega)^2}\sin^2\left(\frac{E_M - E_N \pm \hbar\omega}{2\hbar}t\right).}$$

(e) For light, $V_{ba} = -\wp E_0$ (Eq. 9.34). The rest is as before (Section 9.2.3), leading to Eq. 9.47:

$$\boxed{R_{N\to M} = \frac{\pi}{3\epsilon_0\hbar^2}|\wp|^2\rho(\omega), \text{ with } \omega = \pm(E_M - E_N)/\hbar}$$ (+ sign ⇒ absorption, − sign ⇒ stimulated emission).