Phys 402: Applications of Quantum Mechanics

Homework VIII (Total 2 problems; due 930am, Thursday, March 24, 2016]
[To receive full credits, please show all necessary steps that lead to your answers.]

Read the textbook section 9.1 before starting.

1) Prob. 9.4.

2) Prob. 9.7
(Rabi Oscillation is a popular way to optically control an atomic electronic orbital.)

Prob 9.7.

$$i\dot{C}_a = \frac{V_{ab}}{2} e^{i\omega t} e^{-i\omega_0 t} C_b \; ; \quad i\dot{C}_b = \frac{V_{ba}}{2} e^{-i\omega t} e^{i\omega_0 t} C_a.$$

Differentiate the latter, and substitute in the former:

$$\ddot{c}_b = -i\frac{V_{ba}}{2\hbar}\left[i(\omega_0-\omega)e^{i(\omega_0-\omega)t}c_a + e^{i(\omega_0-\omega)t}\dot{c}_a\right]$$
$$= i(\omega_0-\omega)\left[-i\frac{V_{ba}}{2\hbar}e^{i(\omega_0-\omega)t}c_a\right] - i\frac{V_{ba}}{2\hbar}e^{i(\omega_0-\omega)t}\left[-i\frac{V_{ab}}{2\hbar}e^{-i(\omega_0-\omega)t}c_b\right] = i(\omega_0-\omega)\dot{c}_b - \frac{|V_{ab}|^2}{(2\hbar)^2}c_b.$$

$\frac{d^2c_b}{dt^2} + i(\omega-\omega_0)\frac{dc_b}{dt} + \frac{|V_{ab}|^2}{4\hbar^2}c_b = 0.$ Solution is of the form $c_b = e^{\lambda t}$: $\lambda^2 + i(\omega-\omega_0)\lambda + \frac{|V_{ab}|^2}{4\hbar^2} = 0.$

$$\lambda = \frac{1}{2}\left[-i(\omega-\omega_0) \pm \sqrt{-(\omega-\omega_0)^2 - \frac{|V_{ab}|^2}{\hbar^2}}\right] = i\left[-\frac{(\omega-\omega_0)}{2} \pm \omega_r\right],$$ with ω_r defined in Eq. 9.30.

General solution: $c_b(t) = Ae^{i\left[-\frac{(\omega-\omega_0)}{2}+\omega_r\right]t} + Be^{i\left[-\frac{(\omega-\omega_0)}{2}+\omega_r\right]t} = e^{-i(\omega-\omega_0)t/2}\left[Ae^{i\omega_r t} + Be^{-i\omega_r t}\right],$

or, more conveniently: $c_b(t) = e^{-i(\omega-\omega_0)t/2}\left[C\cos(\omega_r t) + D\sin(\omega_r t)\right].$ But $c_b(0)=0$, so $C=0$:

$$c_b(t) = De^{i(\omega_0-\omega)t/2}\sin(\omega_r t). \quad \dot{c}_b = D\left[i\left(\frac{\omega_0-\omega}{2}\right)e^{i(\omega_0-\omega)t/2}\sin(\omega_r t) + \omega_r e^{i(\omega_0-\omega)t/2}\cos(\omega_r t)\right];$$
$$c_a(t) = i\frac{2\hbar}{V_{ba}}e^{i(\omega-\omega_0)t}\dot{c}_b = i\frac{2\hbar}{V_{ba}}e^{i(\omega-\omega_0)t/2}D\left[i\left(\frac{\omega_0-\omega}{2}\right)\sin(\omega_r t) + \omega_r\cos(\omega_r t)\right].$$ But $c_a(0)=1$:
$$1 = i\frac{2\hbar}{V_{ba}}D\omega_r, \quad \text{or} \quad D = \frac{-iV_{ba}}{2\hbar\omega_r}.$$

$$\boxed{c_b(t) = -\frac{i}{2\hbar\omega_r}V_{ba}e^{i(\omega_0-\omega)t/2}\sin(\omega_r t), \quad c_a(t) = e^{i(\omega-\omega_0)t/2}\left[\cos(\omega_r t) + i\left(\frac{\omega_0-\omega}{2\omega_r}\right)\sin(\omega_r t)\right].}$$

(b)

$P_{a\to b}(t) = |c_b(t)|^2 = \boxed{\left(\frac{|V_{ab}|}{2\hbar\omega_r}\right)^2\sin^2(\omega_r t).}$ The largest this gets (when $\sin^2 = 1$) is $\frac{|V_{ab}|^2/\hbar^2}{4\omega_r^2}$,

and the denominator, $4\omega_r^2 = (\omega-\omega_0)^2 + |V_{ab}|^2/\hbar^2$, exceeds the numerator, so $P \le 1$ (and 1 only if $\omega = \omega_0$).

$$|c_a|^2 + |c_b|^2 = \cos^2(\omega_r t) + \left(\frac{\omega_0-\omega}{2\omega_r}\right)^2\sin^2(\omega_r t) + \left(\frac{|V_{ab}|}{2\hbar\omega_r}\right)^2\sin^2(\omega_r t)$$
$$= \cos^2(\omega_r t) + \frac{(\omega-\omega_0)^2 + (|V_{ab}|/\hbar)^2}{4\omega_r^2}\sin^2(\omega_r t) = \cos^2(\omega_r t) + \sin^2(\omega_r t) = 1. \quad ✓$$

(c) If $\boxed{|V_{ab}|^2 \ll \hbar^2(\omega-\omega_0)^2,}$ then $\omega_r \approx \frac{1}{2}|\omega-\omega_0|$, and $P_{a\to b} \approx \frac{|V_{ab}|^2}{\hbar^2}\frac{\sin^2\left(\frac{\omega-\omega_0}{2}t\right)}{(\omega-\omega_0)^2}$, confirming Eq. 9.28.

(d) $\omega_r t = \pi \Longrightarrow \boxed{t = \pi/\omega_r.}$

and we conclude that for the delta function

$$\boxed{\begin{aligned} c_a(t) &= \begin{cases} 1, & t<0, \\ \cos(|\alpha|/\hbar), & t>0; \end{cases} \\ c_b(t) &= \begin{cases} 0, & t<0, \\ -i\sqrt{\dfrac{\alpha^*}{\alpha}}\sin(|\alpha|/\hbar), & t>0. \end{cases} \end{aligned}}$$

Obviously, $|c_a(t)|^2 + |c_b(t)|^2 = 1$ in both time periods. Finally,

$$\boxed{P_{a\to b} = |b|^2 = \sin^2(|\alpha|/\hbar).}$$

Problem 9.4

(a)

$$\left.\begin{aligned} \text{Eq. 9.10} &\implies \dot{c}_a = -\frac{i}{\hbar}\left[c_a H'_{aa} + c_b H'_{ab}e^{-i\omega_0 t}\right] \\ \text{Eq. 9.11} &\implies \dot{c}_b = -\frac{i}{\hbar}\left[c_b H'_{bb} + c_a H'_{ba}e^{i\omega_0 t}\right] \end{aligned}\right\} \text{ (these are } \textit{exact}\text{, and replace Eq. 9.13).}$$

Initial conditions: $c_a(0) = 1, \quad c_b(0) = 0.$

Zeroth order: $c_a(t) = 1, \quad c_b(t) = 0.$

First order:

$$\begin{cases} \dot{c}_a = -\dfrac{i}{\hbar}H'_{aa} & \implies \boxed{c_a(t) = 1 - \dfrac{i}{\hbar}\displaystyle\int_0^t H'_{aa}(t')\,dt'} \\ \dot{c}_b = -\dfrac{i}{\hbar}H'_{ba}e^{i\omega_0 t} & \implies \boxed{c_b(t) = -\dfrac{i}{\hbar}\displaystyle\int_0^t H'_{ba}(t')e^{i\omega_0 t'}\,dt'} \end{cases}$$

$$|c_a|^2 = \left[1 - \frac{i}{\hbar}\int_0^t H'_{aa}(t')\,dt'\right]\left[1 + \frac{i}{\hbar}\int_0^t H'_{aa}(t')\,dt'\right] = 1 + \left[\frac{1}{\hbar}\int_0^t H'_{aa}(t')\,dt'\right]^2 = 1 \text{ (to first order in } H').$$

$$|c_b|^2 = \left[-\frac{i}{\hbar}\int_0^t H'_{ba}(t')e^{i\omega_0 t'}\,dt'\right]\left[\frac{i}{\hbar}\int_0^t H'_{ab}(t')e^{-i\omega_0 t'}\,dt'\right] = 0 \text{ (to first order in } H').$$

So $|c_a|^2 + |c_b|^2 = 1$ (to first order).

(b)

$$\dot{d}_a = e^{\frac{i}{\hbar}\int_0^t H'_{aa}(t')\,dt'}\left(\frac{i}{\hbar}H'_{aa}\right)c_a + e^{\frac{i}{\hbar}\int_0^t H'_{aa}(t')\,dt'}\dot{c}_a. \quad \text{But } \dot{c}_a = -\frac{i}{\hbar}\left[c_a H'_{aa} + c_b H'_{ab}e^{-i\omega_0 t}\right]$$

Two terms cancel, leaving

$$\begin{aligned} \dot{d}_a &= -\frac{i}{\hbar}e^{\frac{i}{\hbar}\int_0^t H'_{aa}(t')\,dt'}c_b H'_{ab}e^{-i\omega_0 t}. \quad \text{But } c_b = e^{-\frac{i}{\hbar}\int_0^t H'_{bb}(t')\,dt'}d_b. \\ &= -\frac{i}{\hbar}e^{\frac{i}{\hbar}\int_0^t [H'_{aa}(t') - H'_{bb}(t')]dt'}H'_{ab}e^{-i\omega_0 t}d_b, \quad \text{or } \dot{d}_a = -\frac{i}{\hbar}e^{i\phi}H'_{ab}e^{-i\omega_0 t}d_b. \end{aligned}$$

Similarly,

$$\dot{d}_b = e^{\frac{i}{\hbar}\int_0^t H'_{bb}(t')\,dt'}\left(\frac{i}{\hbar}H'_{bb}\right)c_b + e^{\frac{i}{\hbar}\int_0^t H'_{bb}(t')\,dt'}\dot{c}_b. \quad \text{But } \dot{c}_b = -\frac{i}{\hbar}\left[c_b H'_{bb} + c_a H'_{ba}e^{i\omega_0 t}\right].$$

$$= -\frac{i}{\hbar}e^{\frac{i}{\hbar}\int_0^t H'_{bb}(t')\,dt'}c_a H'_{ba}e^{i\omega_0 t}. \quad \text{But } c_a = e^{-\frac{i}{\hbar}\int_0^t H'_{aa}(t')\,dt'}d_a.$$

$$= -\frac{i}{\hbar}e^{\frac{i}{\hbar}\int_0^t [H'_{bb}(t') - H'_{aa}(t')]dt'}H'_{ba}e^{i\omega_0 t}d_a = -\frac{i}{\hbar}e^{-i\phi}H'_{ba}e^{i\omega_0 t}d_a. \quad \text{QED}$$

(c)

Initial conditions: $c_a(0) = 1 \Longrightarrow d_a(0) = 1; \quad c_b(0) = 0 \Longrightarrow d_b(0) = 0.$

Zeroth order: $d_a(t) = 1, \quad d_b(t) = 0.$

First order: $\dot{d}_a = 0 \Longrightarrow d_a(t) = 1 \Longrightarrow \boxed{c_a(t) = e^{-\frac{i}{\hbar}\int_0^t H'_{aa}(t')\,dt'}.}$

$$\dot{d}_b = -\frac{i}{\hbar}e^{-i\phi}H'_{ba}e^{i\omega_0 t} \Longrightarrow d_b = -\frac{i}{\hbar}\int_0^t e^{-i\phi(t')}H'_{ba}(t')e^{i\omega_0 t'}dt' \Longrightarrow$$

$$\boxed{c_b(t) = -\frac{i}{\hbar}e^{-\frac{i}{\hbar}\int_0^t H'_{bb}(t')dt'}\int_0^t e^{-i\phi(t')}H'_{ba}(t')e^{i\omega_0 t'}dt'.}$$

These don't *look* much like the results in (a), but remember, we're only working to *first order* in H', so $c_a(t) \approx 1 - \frac{i}{\hbar}\int_0^t H'_{aa}(t')\,dt'$ (to this order), while for c_b, the factor H_{ba} in the integral means it is *already* first order and hence both the exponential factor in front and $e^{-i\phi}$ should be replaced by 1. Then $c_b(t) \approx -\frac{i}{\hbar}\int_0^t H'_{ba}(t')e^{i\omega_0 t'}dt'$, and we recover the results in (a).

Problem 9.5

Zeroth order: $c_a^{(0)}(t) = a, \quad c_b^{(0)}(t) = b.$

First order:
$$\begin{cases} \dot{c}_a = -\frac{i}{\hbar}H'_{ab}e^{-i\omega_0 t}b \Longrightarrow c_a^{(1)}(t) = a - \frac{ib}{\hbar}\int_0^t H'_{ab}(t')e^{-i\omega_0 t'}dt'. \\ \dot{c}_b = -\frac{i}{\hbar}H'_{ba}e^{i\omega_0 t}a \Longrightarrow c_b^{(1)}(t) = b - \frac{ia}{\hbar}\int_0^t H'_{ba}(t')e^{i\omega_0 t'}dt'. \end{cases}$$

Second order: $\dot{c}_a = -\frac{i}{\hbar}H'_{ab}e^{-i\omega_0 t}\left[b - \frac{ia}{\hbar}\int_0^t H'_{ba}(t')e^{i\omega_0 t'}dt'\right] \Longrightarrow$

$$\boxed{c_a^{(2)}(t) = a - \frac{ib}{\hbar}\int_0^t H'_{ab}(t')e^{-i\omega_0 t'}dt' - \frac{a}{\hbar^2}\int_0^t H'_{ab}(t')e^{-i\omega_0 t'}\left[\int_0^{t'} H'_{ba}(t'')e^{i\omega_0 t''}dt''\right]dt'.}$$

To get c_b, just switch $a \leftrightarrow b$ (which entails also changing the sign of ω_0):

$$\boxed{c_b^{(2)}(t) = b - \frac{ia}{\hbar}\int_0^t H'_{ba}(t')e^{i\omega_0 t'}dt' - \frac{b}{\hbar^2}\int_0^t H'_{ba}(t')e^{i\omega_0 t'}\left[\int_0^{t'} H'_{ab}(t'')e^{-i\omega_0 t''}dt''\right]dt'.}$$
