Phys 402: Applications of Quantum Mechanics

Homework VIII (Total 2 problems; due 930am, Thursday, March 22, 2012]
[To receive full credits, please show all necessary steps that lead to your answers.]

1) Prob. 9.2 (Page 343, textbook. Understand the time dependent perturbation theory, implications.)

2) Prob. 9.3 (You can approximate the delta potential as a square potential in time i.e. H'=0 when t<-T/2, or t>T/2; H'=U/T other wise and take T to be zero. Note that the time integral of H' is equal to U, the strength of the delta potential.)

Problem 9.2

$\dot{c}_a = -\frac{i}{\hbar} H'_{ab} e^{-i\omega_0 t} c_b; \quad \dot{c}_b = -\frac{i}{\hbar} H'_{ba} e^{i\omega_0 t} c_a.$ Differentiating with respect to t:

$$\ddot{c}_b = -\frac{i}{\hbar} H'_{ba} \left[i\omega_0 e^{i\omega_0 t} c_a + e^{i\omega_0 t} \dot{c}_a \right] = i\omega_0 \left[-\frac{i}{\hbar} H'_{ba} e^{i\omega_0 t} c_a \right] - \frac{i}{\hbar} H'_{ba} e^{i\omega_0 t} \left[-\frac{i}{\hbar} H'_{ab} e^{-i\omega_0 t} c_b \right], \text{ or}$$

$$\ddot{c}_b = i\omega_0\dot{c}_b - \frac{1}{\hbar^2}|H'_{ab}|^2 c_b. \quad \text{Let } \alpha^2 \equiv \frac{1}{\hbar^2}|H'_{ab}|^2. \quad \text{Then } \ddot{c}_b - i\omega_0\dot{c}_b + \alpha^2 c_b = 0.$$

This is a linear differential equation with constant coefficients, so it can be solved by a function of the form $c_b = e^{\lambda t}$:

$$\lambda^2 - i\omega_0\lambda + \alpha^2 = 0 \Longrightarrow \lambda = \frac{1}{2}\left[i\omega_0 \pm \sqrt{-\omega_0^2 - 4\alpha^2}\right] = \frac{i}{2}\left(\omega_0 \pm \omega\right), \text{ where } \omega \equiv \sqrt{\omega_0^2 + 4\alpha^2}.$$

The general solution is therefore

$$c_b(t) = Ae^{i(\omega_0+\omega)/2} + Be^{i(\omega_0-\omega)/2} = e^{i\omega_0 t/2}\left(Ae^{i\omega t/2} + Be^{-i\omega t/2}\right), \text{ or}$$

$$c_b(t) = e^{i\omega_0 t/2}\left[C\cos(\omega t/2) + D\sin(\omega t/2)\right]. \quad \text{But } c_b(0) = 0, \quad \text{so } C = 0, \quad \text{and hence}$$

$$c_b(t) = De^{i\omega_0 t/2}\sin(\omega t/2). \quad \text{Then}$$

$$\dot{c}_b = D\left[\frac{i\omega_0}{2}e^{i\omega_0 t/2}\sin(\omega t/2) + \frac{\omega}{2}e^{i\omega_0 t/2}\cos(\omega t/2)\right] = \frac{\omega}{2}De^{i\omega_0 t/2}\left[\cos(\omega t/2) + i\frac{\omega_0}{\omega}\sin(\omega t/2)\right] = -\frac{i}{\hbar}H'_{ba}e^{i\omega_0 t}c_a.$$

$$c_a = \frac{i\hbar}{H'_{ba}}\frac{\omega}{2}e^{-i\omega_0 t/2}D\left[\cos(\omega t/2) + i\frac{\omega_0}{\omega}\sin(\omega t/2)\right]. \quad \text{But } c_a(0) = 1, \quad \text{so } \frac{i\hbar}{H'_{ba}}\frac{\omega}{2}D = 1. \quad \textit{Conclusion:}$$

$$\boxed{\begin{aligned} c_a(t) &= e^{-i\omega_0 t/2}\left[\cos(\omega t/2) + i\frac{\omega_0}{\omega}\sin(\omega t/2)\right], \\ c_b(t) &= \frac{2H'_{ba}}{i\hbar\omega}e^{i\omega_0 t/2}\sin(\omega t/2), \end{aligned}} \quad \text{where } \boxed{\omega \equiv \sqrt{\omega_0^2 + 4\frac{|H'_{ab}|^2}{\hbar^2}}.}$$

$$\begin{aligned} |c_a|^2 + |c_b|^2 &= \cos^2(\omega t/2) + \frac{\omega_0^2}{\omega^2}\sin^2(\omega t/2) + \frac{4|H'_{ab}|^2}{\hbar^2\omega^2}\sin^2(\omega t/2) \\ &= \cos^2(\omega t/2) + \frac{1}{\omega^2}\left(\omega_0^2 + 4\frac{|H'_{ab}|^2}{\hbar^2}\right)\sin^2(\omega t/2) = \cos^2(\omega t/2) + \sin^2(\omega t/2) = 1. \quad \checkmark \end{aligned}$$

Problem 9.3

This is a tricky problem, and I thank Prof. Onuttom Narayan for showing me the correct solution. The safest approach is to represent the delta function as a sequence of rectangles:

$$\delta_\epsilon(t) = \begin{Bmatrix} (1/2\epsilon), & -\epsilon < t < \epsilon, \\ 0, & \text{otherwise.} \end{Bmatrix}$$

Then Eq. 9.13 $\Rightarrow$

$$\begin{cases} t < -\epsilon: & c_a(t) = 1, \ c_b(t) = 0, \\ t > \epsilon: & c_a(t) = a, \ c_b(t) = b, \\ -\epsilon < t < \epsilon: & \begin{cases} \dot{c}_a = -\frac{i\alpha}{2\epsilon\hbar}e^{-i\omega_0 t}c_b, \\ \dot{c}_b = -\frac{i\alpha^*}{2\epsilon\hbar}e^{i\omega_0 t}c_a. \end{cases} \end{cases}$$

In the interval $-\epsilon < t < \epsilon$,

$$\frac{d^2c_b}{dt^2} = -\frac{i\alpha^*}{2\epsilon\hbar}\left[i\omega_0 e^{i\omega_0 t}c_a + e^{i\omega_0 t}\left(\frac{-i\alpha}{2\epsilon\hbar}e^{-i\omega_0 t}c_b\right)\right] = -\frac{i\alpha^*}{2\epsilon\hbar}\left[i\omega_0\frac{i2\epsilon\hbar}{\alpha^*}\frac{dc_b}{dt} - \frac{i\alpha}{2\epsilon\hbar}c_b\right] = i\omega_0\frac{dc_b}{dt} - \frac{|\alpha|^2}{(2\epsilon\hbar)^2}c_b.$$

Thus c_b satisfies a homogeneous linear differential equation with constant coefficients:

$$\frac{d^2c_b}{dt^2} - i\omega_0\frac{dc_b}{dt} + \frac{|\alpha|^2}{(2\epsilon\hbar)^2}c_b = 0.$$

Try a solution of the form $c_b(t) = e^{\lambda t}$:

$$\lambda^2 - i\omega_0\lambda + \frac{|\alpha|^2}{(2\epsilon\hbar)^2} = 0 \Rightarrow \lambda = \frac{i\omega_0 \pm \sqrt{-\omega_0^2 - |\alpha|^2/(\epsilon\hbar)^2}}{2},$$

or

$$\lambda = \frac{i\omega_0}{2} \pm \frac{i\omega}{2}, \text{ where } \omega \equiv \sqrt{\omega_0^2 + |\alpha|^2/(\epsilon\hbar)^2}.$$

The general solution is

$$c_b(t) = e^{i\omega_0 t/2}\left(Ae^{i\omega t/2} + Be^{-i\omega t/2}\right).$$

But

$$c_b(-\epsilon) = 0 \Rightarrow Ae^{-i\omega\epsilon/2} + Be^{i\omega\epsilon/2} = 0 \Rightarrow B = -Ae^{-i\omega\epsilon},$$

so

$$c_b(t) = Ae^{i\omega_0 t/2}\left(e^{i\omega t/2} - e^{-i\omega(\epsilon+t/2)}\right).$$

Meanwhile

$$\begin{aligned} c_a(t) &= \frac{2i\epsilon\hbar}{\alpha^*}e^{-i\omega_0 t}\dot{c}_b = \frac{2i\epsilon\hbar}{\alpha^*}e^{-i\omega_0 t/2}A\left[\frac{i\omega_0}{2}\left(e^{i\omega t/2} - e^{-i\omega(\epsilon+t/2)}\right) + \frac{i\omega}{2}\left(e^{i\omega t/2} + e^{-i\omega(\epsilon+t/2)}\right)\right] \\ &= -\frac{\epsilon\hbar}{\alpha^*}e^{-i\omega_0 t/2}A\left[(\omega+\omega_0)e^{i\omega t/2} + (\omega-\omega_0)e^{-i\omega(\epsilon+t/2)}\right]. \end{aligned}$$

But $\quad c_a(-\epsilon) = 1 = -\frac{\epsilon\hbar}{\alpha^*}e^{i(\omega_0-\omega)\epsilon/2}A\left[(\omega+\omega_0) + (\omega-\omega_0)\right] = -\frac{2\epsilon\hbar\omega}{\alpha^*}e^{i(\omega_0-\omega)\epsilon/2}A, \quad$ so $\quad A = -\frac{\alpha^*}{2\epsilon\hbar\omega}e^{i(\omega-\omega_0)\epsilon/2}.$

$$\begin{aligned} c_a(t) &= \frac{1}{2\omega}e^{-i\omega_0(t+\epsilon)/2}\left[(\omega+\omega_0)e^{i\omega(t+\epsilon)/2} + (\omega-\omega_0)e^{-i\omega(t+\epsilon)/2}\right] \\ &= e^{-i\omega_0(t+\epsilon)/2}\left\{\cos\left[\frac{\omega(t+\epsilon)}{2}\right] + i\frac{\omega_0}{\omega}\sin\left[\frac{\omega(t+\epsilon)}{2}\right]\right\}; \\ c_b(t) &= -\frac{i\alpha^*}{2\epsilon\hbar\omega}e^{i\omega_0(t-\epsilon)/2}\left[e^{i\omega(t+\epsilon)/2} - e^{-i\omega(t+\epsilon)/2}\right] = -\frac{i\alpha^*}{\epsilon\hbar\omega}e^{i\omega_0(t-\epsilon)/2}\sin\left[\frac{\omega(t+\epsilon)}{2}\right]. \end{aligned}$$

Thus

$$a = c_a(\epsilon) = e^{-i\omega_0\epsilon}\left[\cos(\omega\epsilon) + i\frac{\omega_0}{\omega}\sin(\omega\epsilon)\right], \quad b = c_b(\epsilon) = -\frac{i\alpha^*}{\epsilon\hbar\omega}\sin(\omega\epsilon).$$

This is for the rectangular pulse; it remains to take the limit $\epsilon \to 0$: $\omega \to |\alpha|/\epsilon\hbar$, so

$$a \to \cos\left(\frac{|\alpha|}{\hbar}\right) + i\frac{\omega_0\epsilon\hbar}{|\alpha|}\sin\left(\frac{|\alpha|}{\hbar}\right) \to \cos\left(\frac{|\alpha|}{\hbar}\right), \quad b \to -\frac{i\alpha^*}{|\alpha|}\sin\left(\frac{|\alpha|}{\hbar}\right),$$

and we conclude that for the delta function

$$\boxed{\begin{aligned} c_a(t) &= \begin{cases} 1, & t < 0, \\ \cos(|\alpha|/\hbar), & t > 0; \end{cases} \\ c_b(t) &= \begin{cases} 0, & t < 0, \\ -i\sqrt{\dfrac{\alpha^*}{\alpha}}\sin(|\alpha|/\hbar), & t > 0. \end{cases} \end{aligned}}$$

Obviously, $|c_a(t)|^2 + |c_b(t)|^2 = 1$ in both time periods. Finally,

$$\boxed{P_{a\to b} = |b|^2 = \sin^2(|\alpha|/\hbar).}$$